STRING FRAME AND EINSTEIN FRAME

A SAMMLUNG OF PAPERS

FOR HIS OWN STUDY

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Are the String and Einstein Frames Equivalent?

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Abstract

The low energy physics as predicted by strings can be expressed in two (conformally related) different variables, usually called *frames*. The problem is raised as to whether it is physically possible in some situations to tell one from the other.

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1 Introduction

In the low energy (that is, wavelengths much longer than the string scale, $l_s \equiv (\alpha')^{1/2}$) limit all string theories predict at least a common sector of massless fields (in any dimension): one associated with the *graviton* $g_{\mu\nu}$, another with a scalar field ϕ, called the *dilaton*[2], which is related to the string coupling constant, g_s in the sense that a *constant* variation of the dilaton, $\delta\phi$ produces a corresponding variation in the coupling constant $\frac{\delta g_s}{g_s} = \delta\phi$, and a two- index antisymmetric field, $b_{\mu\nu}$, called the Kalb-Ramond field, which is often associated to axions in four dimensional compactifications.

The coupling of these fields to the embeddings of the two-dimensional world sheet of the string, Σ in the spacetime M, $x^\mu(\sigma,\tau)$ is given by the two-dimensional nonlinear sigma model which in the conformal gauge reads:

$$S_2 = \frac{1}{4\pi l_s^2}\int d^2\sigma \left[g_{\mu\nu}\partial_a x^\mu \partial_b x^\nu \eta^{ab} + i b_{\mu\nu}\partial_a x^\mu \partial_b x^\nu \epsilon^{ab} + l_s^2 R^{(2)}\phi(x)\right] \tag{1}$$

(where $\sigma^a \equiv (\sigma^1,\sigma^2) \equiv (\sigma,\tau)$, η^{ab} represents the two-dimensional flat metric, and ϵ^{ab} stands for the two-dimensional Levi-Civita symbol, whereas $R^{(2)}$ stands for the two-dimensional scalar curvature).

In order for self-consistency, (i.e., two-dimensional conformal invariance, which entails the vanishing of the corresponding β-functions) these backgrounds have to obey some equations which, to the lowest order in l_s^2 can be derived from a n-dimensional action principle:

$$S_n \equiv \int d(vol)_n \frac{1}{2\kappa_n^2}\, e^{-2\phi}\left[\, R - \frac{n-26}{3l_s^2} - \frac{1}{12}H_{abc}H^{abc} + 4(\nabla\phi)^2\right] \tag{2}$$

(where $H \equiv db$ is the field strength of the Kalb-Ramond field, i.e., $H_{abc} = \partial_{[a}b_{bc]}$, and $d(vol)_n \equiv \sqrt{g}d^n x$, whereas κ_n stands for the n-dimensional Planck's constant).

The mixing between the dilaton and the graviton in the kinetic term can be avoided

[2] Which we shall choose as dimensionless.

with a field redefinition consistent in a Weyl transformation:

$$g^E_{\mu\nu} \equiv e^{\frac{4(\phi_0-\phi)}{n-2}} g_{\mu\nu} \tag{3}$$

(where $\phi_0 \equiv < \phi >$ is the unknown dilaton vacuum expectation value, which is different in principle from its asymptotic value at infinity , ϕ_∞) leading to the spacetime effective action written in what is usually called the *Einstein Frame* (while we say that the former action (2) was written in the *String Frame*).

$$S^E_n \equiv \int d(vol)_n \frac{1}{2\kappa_n^2} \left[R - \frac{n-26}{3l_s^2} e^{\frac{4\phi_E}{n-2}} - \frac{1}{12} e^{-\frac{8\phi_E}{n-2}} H_{abc} H^{abc} - \frac{4}{n-2} (\nabla \phi_E)^2 \right] \tag{4}$$

and we have defined $\phi_E \equiv \phi - \phi_0$, in such a way that $< \phi_E >= 0$.

There are other frames associated with topological defects, like D-branes, which differ in the power of the exponential of the dilaton (because they appear at different order in the string coupling constant g_s). They can be easily included in the framework of our discussion, however.

Although it is not known what is the general form of the higher order (sigma model) corrections, it has been conjectured in [6] that it would have the general form [3] in the string frame:

$$\begin{aligned} S_n = \int d(vol)_n \Big[& \frac{B_g(\phi)}{l_s^{n-2}} R + \frac{B_\phi(\phi)}{l_s^{n-2}} (4\nabla^2 \phi - 4(\nabla\phi)^2) \\ & - \frac{B_f(\phi)}{l_s^{n-4}} \frac{k}{4} tr F^2 - B_\psi(\phi) \bar{\psi} \not{\nabla} \psi - B_m(\phi) m \bar{\psi} \psi \Big] \end{aligned} \tag{5}$$

The dilaton dressing functions $B_i(\phi)$ are unknown, and we have indicated some of the gauge fields and fermion fields present, as well as a typical mass term, which will be needed in further considerations. The Kac-Moody level k is anumerical constant. The term involving the dilaton kinetic energy can be rewritten upon partial integration [4] as:

$$- \frac{4}{l_s^{n-2}} (B'_\phi + B_\phi)(\nabla\phi)^2 \tag{6}$$

[3] This form is in any case general enough to include all internediate frames associated to different *branes*

[4] Which is only valid when the dilaton field vanishes at infinity fast enough

The Einstein frame will now be defined through the Weyl transformation:

$$g_{\mu\nu} \equiv \frac{(l_s m_p)^2}{B_g^{2/(n-2)}} g^E_{\mu\nu} \tag{7}$$

(where m_p is the Plack mass) and a redefinition of the dilaton:

$$\phi = \int^{\phi_E} \left[\frac{n-1}{4} (\frac{B'_g}{B_g})^2 + (n-2) \frac{B'_\phi + B_\phi}{B_g} \right]^{-1/2} \tag{8}$$

Fermions are redefined as well:

$$\psi = \frac{B_g^{(n-1)/(2(n-2))}}{(l_s m_p)^{(n-1)/2} B_\psi^{1/2}} \psi_E \tag{9}$$

This leads to the Einstein frame effective action in which the Einstein Hilbert term is not mixed with the dilaton:

$$S^E_n = \int d(vol)^E_n \left[\frac{1}{m_p^{2-n}} R_E - \frac{4}{(n-2) m_p^{2-n}} (\nabla \phi_E)^2 \right.$$
$$\left. - \frac{B_f(\phi)}{B_g^{(n-4)/(n-2)} m_p^{4-n}} \frac{k}{4} tr F^2 - \bar{\psi}_E \not{\nabla} \psi_E - B_m(\phi) m \frac{l_s m_p}{B_\psi B_g^{1/(n-2)}} \bar{\psi}_E \psi_E \right] \tag{10}$$

The coefficient of the dilaton kinetic energy is conventional, and can be fixed at will. In the new Frame there are extra terms of Yukawa type bewteen the dilaton and the fermions which have been left implicit. It is worth noticing that masses refered to the natural unit of mass in the string frame, i.e., $m_s \equiv \frac{B_g^{1/(n-2)}}{l_s}$ do not change in the Einstein frame (where the unit of mass is m_p), provided we divide as well by the kinetic energy factor (i.e., B_ψ). This means that, in a precise sense, properly renormalized masses are invariant under change of frame.

The two frames are usually considered as completely equivalent for describing the physics of the massless modes of the string. This fact (although occasionally questioned in the literature, (cf.[3],[4],[8],[7])), seems hardly arguable, at least as long as the (classical) functions involved are smooth; given any solution of the system of differential equations which encode the equations of motion in one frame, there exists a corresponding solution in the other frame.

The equations of motion for the metric and the dilaton in the String Frame read:

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{B_g}{l_s^{n-2}} R_{\mu\nu} - \frac{4}{l_s^{n-2}}(B'_\phi + B_\phi)\nabla_\mu\phi\nabla_\nu\phi - \frac{B_f}{l_s^{n-4}}\frac{k}{2} tr F_{\mu\alpha}F_\nu^\alpha - B_\psi\bar{\psi}\gamma_\mu\nabla_\nu\psi$$
$$-\frac{1}{2}g_{\mu\nu}(\frac{B_g}{l_s^{n-2}}R - \frac{4(B_\phi + B'_\phi)}{l_s^{n-2}}(\nabla\phi)^2 - \frac{B_f}{l_s^{n-4}}\frac{k}{4}trF^2 - B_\psi\bar{\psi}\ \not{\nabla}\psi - mB_m\bar{\psi}\psi) = 0 \quad (11)$$

and

$$\frac{\delta S}{\delta\phi} = \frac{B'_\phi}{l_s^{n-2}}R + \frac{4}{l_s^2}(B''_\phi + B'_\phi)(\nabla\phi)^2 - \frac{kB'_f}{4l_s^{n-4}}trF^2$$
$$-B'_\psi\bar{\psi}\ \not{\nabla}\psi + \frac{8}{l_s^{n-2}}(B'_\phi + B_\phi)\nabla^2\phi - mB'_m\bar{\psi}\psi = 0. \quad (12)$$

The equivalence of the equations of motion in the two frames has been worked out in detail in ([5]). Let us now however discuss in turn some delicate issues.

2 Dualities

There is now a certain amount of evidence for different kinds of symmetries between different string theories (See for example [11]). The two more important ones are *S-duality* and *T-duality*. We shall say that two (not neccessarily different) theories, T_1 and T_2 are T-dual, when T_1 compactified at large Kaluza-Klein volume is physically equivalent to T_2 at small Kaluza-Klein volume. If we call t the modulus associated to global variations of the Kaluza-Klein volume, by $Vol \sim e^t$, this implies a relationship of the general form

$$t(1) \ = \ -t(2)\ . \quad (13)$$

This symmetry can be proven true when there is an isometry in the spacetime manifold whose Killing vector is written in adapted coordinates as $\frac{\partial}{\partial x^0}$ by several means ([1],[10])and is such that the string metric transforms in a simple way, to wit:

$$\tilde{g}_{00} \ = \ \frac{1}{g_{00}}\ ,$$
$$\tilde{g}_{0i} \ = \ \frac{b_{0i}}{g_{00}}\ , \qquad \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}}\ ,$$

$$\begin{aligned} \tilde{g}_{ij} &= g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}} \,, \\ \tilde{b}_{ij} &= b_{ij} - \frac{g_{0i}b_{0j} - g_{0j}b_{0i}}{g_{00}} \,. \end{aligned} \tag{14}$$

S-duality, on the other hand, refers to the equivalence of T_1 at small coupling with T_2 at large coupling. Given the relationship we already mentioned between the dilaton and the string coupling, it demands for the dilaton something like

$$\phi(1) \;=\; -\phi(2) \,, \tag{15}$$

and, by definition, lies beyond the possibilities of verification by means of perturbation theory. For IIB strings, the (modified, cf. ref. ([12]) Einstein metric is inert under this transformation,

$$\tilde{g}^E_{\mu\nu} = g^E_{\mu\nu} \tag{16}$$

We then se that each frame seems to be most appropiate depending on which symmetry we believe to be the most fundamental.

3 The definition of the vacuum state

Insofar as

$$T^{(matt)}_{\mu\nu} = 0 \tag{17}$$

we always have a *vacuum* solution of the equations og motion:

$$\begin{aligned} \phi &= 0 \\ g_{\mu\nu} &= \eta_{\mu\nu} \end{aligned} \tag{18}$$

On the other hand, it has been proven in ref. [6] that under certain hypothesis (mainly the equality of all dressing functions B_i), the cosmological evolution attracts the dilaton towards the point where the dressed masses are stationary

$$\frac{\partial m_E}{\partial \phi}|_{\phi=\phi_0} = 0 \tag{19}$$

It has also been argued that the existence of several different minima is probably incompatible with present bounds on the equivalence principle (namely a relative difference in acceleration $\frac{\Delta a}{a} \leq 10^{-13}$ [6])). Although it is known that in general the hypothesis of reference ([6]) are not fulfilled, there are actual string models where this mechanism is automatic ([2]).

4 The principle of equivalence

Were to be strings the probes of the metric, it is obvious that the most natural frame would be the String Frame. But in most classical experiments, the metric is detected through its effect on classical test particles, which describe geodesics of the spacetime metric. At a higher level of precision, the geodesic deviation equation, gives direct information of the Riemann tensor.

Let us now review how the concept of *particle* is recovered from the concept of *field*. The latter is written as a formal (WKB) series

$$\phi = e^{\frac{1}{\epsilon}\sum \epsilon^n \phi_n} \tag{20}$$

The Klein-Gordon equation then gives, in the eikonal approximation (actually, to the dominant order $\frac{1}{\epsilon^2}$)

$$k^2 = -m^2 \tag{21}$$

where the mass has to be considered as $o(\epsilon^{-2})$, and

$$k_\mu \equiv \nabla_\mu \phi_0 \tag{22}$$

This implies that the flow lines defined by the congruence k^μ are geodesic, since

$$0 = \nabla_\rho (k^2) = 2k^\mu \nabla_\rho k_\mu = 2k^\mu \nabla_\mu k_\rho \tag{23}$$

where the last step is justified since the vector k^ρ is itself a gradient.

Let us now consider a scalar field other than the dilaton (which remains massless to all orders in perturbation theory), with Lagrange density:

$$L \equiv -\frac{1}{2}(B_\chi(\nabla\chi)^2 + m^2 B_m \chi^2) \tag{24}$$

The equations of motion in the eikonal approximation now yield

$$B_\chi k^2 = -m^2 B_m \tag{25}$$

which violates the principle of equivalence unless $B_m = B_\chi$, but does not distinguish qualitatively (although it does it quantitatively) between different frames.

That is, particles will propagate along geodesics of that metric (if any) such that $B_\chi = B_m$. In addition, the dressing factors B_χ will depend generically on the particle considered, and so will depend the trajectories, and it is this fact which violates the equivalence principle.

5 The fluid approximation

The situation is perhaps less clear when both the metric and the dilaton are singular in one frame, but the metric is regular in the other. This clearly changes the physics of *test particles* propagating in the physical spacetime.

Let us now point out a related, but simpler, situation.

Imagine that matter (that is all fields except gravitation itself)is such that its energy-momentum tensor corresponds to a perfect fluid,

$$T_{\mu\nu} \equiv (\rho + p)\ u_\mu u_\nu + p g_{\mu\nu} \tag{26}$$

Then the question is: will the same matter still behave as a perfect fluid in the other frame? This is a meaningful question, because matter is almost always considered of such a form in cosmological investigations.

The general conditions for a fluid description to be valid consist in demanding that the wavelength (as measured in a Local Inertial Frame, LIF, defined by a *vielbein*, $e_a^\mu \partial_\mu$, $a = 0, \ldots, n-1$) should be much less than both the macroscopical length of the wavepacket, l, and the scale of variation of Riemann's tensor, r.

$$\lambda << \ min\ (r, l) \tag{27}$$

Now, when changing frames, quantities in the LIF obviously do not change,(neglecting the new interacions introduced through the change of frame) whereas lengths scale as

$$l_S = l_E \frac{l_s m_p}{B_g^{1/(n-2)}} \tag{28}$$

(Where an average value for the dilaton field in the region considered is implicitly assumed). In the case of Riemann's tensor, there are in addition extra terms proportional to the square of the first derivative and to the second derivative of the dilaton field, which could dominate for a rapidly fluctuating dilaton.

A sufficient condition for such a hydrodynamic fluid description is to be at thermodynamic equilibrium. It is in turn clear that a neccessary condition for it (it is less clear whether it will be *sufficient*) is that the mean free time between collisions should be smaller than Hubble's time

$$\tau << H^{-1} \tag{29}$$

The mean free time, in turn, is related to quantities computed in a LIF: basically $\tau \sim (\rho v \sigma)^{-1}$, where ρ is the average density of particles, v is the average velocity, and σ is the total cross section (again, neglecting the new interactions).

The scale factor, on the other hand, is a global quantity, and, as such, scales as above

$$R_S = R_E \frac{l_s m_p}{B_g^{1/(n-2)}} \tag{30}$$

in such a way that

$$H_S \equiv \frac{\dot{R}_S}{R_S} = H_E - \frac{1}{n-2} \frac{B_g'}{B_g} \dot{\phi} \tag{31}$$

The crutial quantity now is the dimensionless quotient

$$\frac{B_g' \dot{\Phi}}{(n-2) B_g H_E} \tag{32}$$

If it is small, equilibrium in both frames is equivalent.

But if it is large (corresponding physically to a wildly fluctuating dilaton *or* to the vicinity of a place in which the dilaton itself is singular), then equilibrium in Einstein's frame does not guarantee equilibrium in the string frame.

It is worth remarking that even for a time independent dilaton (which does not spoil the equilibrium condition) the preceding formula (28) indicates that for large dilatons averaged (corresponding to large g_s), so that $B_g << 1$ and correspondingly, $l_s >> l_E$, matter enjoying a fluid description in the String Frame does not neccessarily do so in the Einstein Frame. For large *negative* averaged dilaton couplings (corresponding to small g_s) the converse is true: a fluid description in the Einstein Frame does not guarantee a fluid description in String Frame).

6 Conclusions

There is in our opinion no doubt of the equivalence of all frames for the description of the gravitational effects of string theories at a basic level, at least when all functions involved are smooth.

When this is not the case, the solution depends on what is the quantum resolution of the classical gravitational singularities. The symmetries of string theory (T-duality, in particular) suggest that there is, in a sense, a minimal measurable length, but the issue is far from settled; it could be, in particular, that certain particle-like topological defects, known as $D0$ branes, could probe shorter lengths.

A different question is what is the classical metric felt by a particular probe (usually, a *test particle*). Here, again, strings give a unique answer (depending on the probe

used), which seems difficult to reconcile *a priori* with existing bounds on violations of the equivalence principle. A detailed comparison is however difficult owing to our limited understanding of the dynamics of the dilaton as well as other scalars in the string spectrum.

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Little String Theory at a TeV

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We propose a framework where the string scale as well as all compact dimensions are at the electroweak scale $\sim \mathrm{TeV}^{-1}$. The weakness of gravity is attributed to the small value of the string coupling $g_s \sim 10^{-16}$, presumably a remnant of the dilaton's runaway behavior, suggesting the possibility of a common solution to the hierarchy and dilaton-runaway problems. In spite of the small g_s, in type II string theories with gauge interactions localized in the vicinity of NS5-branes, the standard model gauge couplings are of order one and are associated with the sizes of compact dimensions. At a TeV these theories exhibit higher dimensional and stringy behavior. The models are holographically dual to a higher dimensional non-critical string theory and this can be used to compute the experimentally accessible spectrum and self-couplings of the little strings. In spite of the stringy behavior, gravity remains weak and can be ignored at collider energies. The Damour-Polyakov mechanism is an automatic consequence of our scenario and suggests the presence of a massless conformally-coupled scalar, leading to potentially observable deviations from Einstein's theory, including violations of the equivalence principle.
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1. Introduction

An obstacle to building a unified theory of all forces is the enormous disparity between the gravitational and other forces, commonly referred to as the hierarchy problem. In the standard framework of particle physics this is answered at the expense of postulating an enormous energy desert separating the gravitational from the electroweak scale [1]. The supersymmetric version of this picture [2], called the supersymmetric standard model, has had a quantitative success: the unification prediction of the value of the weak mixing angle [2], subsequently confirmed by the LEP and SLC experiments. This makes it tempting to believe in the unification of the non-gravitational forces at a large energy scale $\sim 10^{16}$ GeV. Nevertheless, this picture leaves many fundamental questions unanswered. There are 125 parameters in the supersymmetric standard model that remain unexplained. These include the masses of the three generations of particles and, above all, the incredible smallness of the cosmological constant. This suggests that there are enormous gaps in our understanding of Nature at low energies and that perhaps we will need a radical revision of our fundamental view of the world at low energies, at least with respect to gravity.

On the other hand, string theory provides the only known framework for quantizing gravity. The cost is to replace our fundamental concept of point particles by extended objects whose quantum consistency requires the existence of extra dimensions. One of the important consequences of the recent theoretical progress on the non-perturbative dynamics of string theories is that the string and compactification scales are not necessarily tied to the four-dimensional Planck mass [3,4,5]. This opens the exciting possibility that string physics may become relevant at much lower energies with spectacular new effects in future accelerators.

Such a possibility can also be used to explain the hierarchy problem, motivated by the following string theoretic expression for the four-dimensional (4d) Planck mass [6]:

$$M_P^2 = \frac{1}{g_s^2} M_s^8 V_6 \ , \tag{1.1}$$

where g_s is the string coupling, M_s the string scale and V_6 the volume of the six-dimensional internal space. This relation shows that it is possible that there is only one fundamental scale in the universe, the electroweak scale $\sim$ TeV, where all forces of nature, including gravity, unify and therefore $M_s \sim$ TeV. Then the enormity of the Planck scale can be accounted for in two distinct ways:

(1) A non-stringy way [6]:
This is realized if V_6 is enormously larger than the fundamental scale while keeping g_s of order unity. In order to make such large dimensions consistent with observations, gauge interactions should be localized on branes transverse to them. A natural framework for realizing this scenario is type I string theory with the Standard Model (SM) confined on a collection of Dp-branes. Perturbative calculability requires the $p-3$ longitudinal dimensions to be compactified near the (TeV) string scale, while the $9-p$ transverse dimensions should be much larger in order to account for the observed weakness of gravitational interactions.

While this scenario can be naturally imbedded in type I string theories, it does not require string theory for its implementation at low energies, below the (TeV) scale of quantum gravity. The physical mechanism is the dilution of the strength of gravity by spreading it into extra dimensions, which could have been invented by Gauss two centuries ago. The hierarchy problem now turns into one of explaining dynamically the large magnitude of V_6.

(2) A stringy way:
This is realized by taking V_6 to be of the order of the fundamental scale $\sim \mathrm{TeV}^{-6}$ and attributing the enormity of the Planck mass to a tiny $g_s \sim 10^{-16}$ [7] (see also [5]). [2] The hierarchy problem is now equivalent to understanding the smallness of g_s, or equivalently the large value of the dilaton field in our universe.

Starting with [6], possibility (1) has been explored extensively in the last three years. Our objective in this paper is to study the second logical possibility (2) which is stringy in nature, at least in the sense that it involves g_s, and gives a new perspective to the hierarchy and other problems in physics. A fundamental question now becomes whether a string theory with such a small g_s can contain the ordinary gauge interactions whose dimensionless couplings are of order unity. Fortunately the answer is yes in the context of special type II string theories whose gauge interactions are localized in the vicinity of NS5-branes, which we will utilize here. In these theories gauge couplings are given by the geometric sizes of new dimensions and are non-vanishing even if g_s vanishes.

In the limit of vanishing g_s [8] one obtains a theory without gravity, the so-called Little String Theory (LST); it was introduced in [9,8] (see also [10],[11], and for a review

[2] Actually, the value of g_s is more likely 10^{-14} [7], corresponding in eq. (1.1) to a volume $V_6 \simeq (2\pi)^6$ of a toroidal compactification with all radii fixed at the string length.

see [12] and references therein). LST is a partial string theory; although it does not include gravity, it has string excitations and therefore is not a normal local field theory. It is an intermediate logical possibility between full-fledged string theory and field theory. The main objective of our paper is to point out that this intermediate possibility can be realized at the experimentally accessible energy of $\sim$ TeV and give us an alternate way to address the hierarchy problem which connects it with the dilaton-runaway problem. This therefore interpolates between the TeV-strings framework [6], which has full string theory at a TeV, and other field-theoretic possibilities for TeV physics such as supersymmetry, technicolor or warped compactifications [13].

We propose three closely related frameworks for building realistic theories with little strings at a TeV. Their common feature is the existence of closed little strings with $\sim$ TeV tension, whose self interaction and spectroscopy can be computed in some cases. In addition there can be string excitations of ordinary particles, with either the same or different tension, as well as KK and winding modes associated with $\sim$ TeV^{-1} size dimensions.

An unexpected bonus of the framework is that the Damour-Polyakov mechanism, based on the universality of the dilaton coupling functions and normally considered improbable, becomes automatic. It may lead to small but potentially observable deviations from the equivalence principle [14].

In section 2, we discuss mass scales and couplings in type II string theories, and define our general framework. In section 3, we recall the possible descriptions of little string theories and in particular the double scaling limit which defines a sensible perturbation theory. In section 4, we touch on some basic phenomenological consequences of the framework. Section 5 addresses the hierarchy problem and suggests ways in which the dilaton field can have a naturally large value in our universe. In section 6, we remark on a possible implication of our framework for the cosmological constant problem and other topics.

2. Mass Scales and Couplings

In every perturbative string theory, gravity arises from closed strings that propagate in ten dimensions. As a result, the 4d Planck mass is given by eq. (1.1). Here, all internal dimensions are taken to be larger than the string length $l_s \equiv M_s^{-1}$ by a suitable choice of T-dualities; thus, in this convention, all closed string winding modes are heavier than the string scale. The strength of gravity at energies above all compactification scales and

below the string scale, $V_6^{-1/6} < E < M_s$, is determined by the ten-dimensional Planck mass

$$M_{10}^8 = \frac{M_P^2}{V_6} = \frac{M_s^8}{g_s^2} \ , \tag{2.1}$$

which can be obtained by summing over all KK graviton excitations yielding a suppression proportional to $(E/M_{10})^8 = (E/M_s)^8 g_s^2$. It follows that at energies of order the string scale gravitational interactions are controlled by the string coupling g_s.

On the other hand, in type II theories non-abelian gauge interactions arise non-perturbatively localized on (Neveu-Schwarz) NS5-branes, corresponding in the simplest case to D-branes stretched between the NS5-branes. In a T-dual picture, non-abelian fields in (supersymmetric) type IIA (IIB) theories emerge from D2 (D3) branes wrapping around collapsing 2-cycles of the compactification manifold [15]. Such 2-cycles are localized in a subspace of dimension 4 that defines (upon T-duality) the transverse position of the NS5-branes where gauge interactions are confined. Furthermore, the four-dimensional Yang-Mills (YM) coupling is determined by the geometry of the two-dimensional compact space along the NS5-branes, independently of the value of the string coupling g_s.

For instance, let us consider a stack of NS5-branes extended in the directions $X^{0,1,2,3,4,5}$, where $X^{0,1,2,3}$ define our 1+3 dimensional spacetime. The extra two longitudinal directions $X^{4,5}$ are compactified on a rectangular torus T^2 with radii $R_{4,5}$, while the four transverse directions $X^{6,7,8,9}$ are compactified on a manifold with size R_t. The four-dimensional gauge coupling is then given by

$$\begin{aligned} &\text{type IIA :} && g_{YM}^2 = \frac{R_4}{R_5} \ , \\ &\text{type IIB :} && g_{YM}^2 = \frac{l_s^2}{R_4 R_5} \ . \end{aligned} \tag{2.2}$$

In summary, in type II theories gravitational interactions are controlled by the string coupling, while gauge interactions are governed by geometrical moduli along the 5-branes where they are confined. This is in contrast with type I theories, where the string coupling determines also the strength of gauge interactions confined on D-branes and is therefore fixed to be of order one [3]; thus, in type I theories gravitational interactions become strong at the string scale.

It follows that the type II string scale can be lowered at the TeV scale without introducing extra large transverse dimensions, but instead a tiny string coupling to account

[3] Here we drop factors of π and for numerical estimates we use $g_{YM} \simeq 0.1$.

for the hierarchy M_s/M_P [7]. In this case, the physics around the string scale is described approximately by a theory without gravity obtained in the weak coupling limit $g_s \to 0$ [8]. This theory, which is defined in the limit of coincident NS5-branes with vanishing string coupling, is called little string theory (LST) [8-12].

This theory lives in six dimensions and contains two sectors. The charged (non-abelian) sector confined on the NS5-branes and a neutral sector of closed fundamental strings trapped in the vicinity of the NS5-branes. In section 3, we review the main properties of these theories, while in section 4 we discuss their phenomenological consequences when the fundamental string scale is in the TeV region.

One may ask the question whether a tiny string coupling can be described alternatively, via some duality, in terms of large dimensions in the context of M-theory. Indeed, it was shown that the weakly coupled type II string compactified for instance on $K3 \times T^2$, with all compactified dimensions of string size, provides a dual description to the strongly coupled heterotic theory compactified on $T^4 \times T^2$ with the four dimensions of T^4 having the heterotic string size $l_H \sim M_P^{-1}$ while the two dimensions of T^2 being much larger, of the order of the type II string length [7]. If the type II string scale $M_s \sim$ TeV, its string coupling $g_s \sim M_s/M_P \simeq 10^{-16}$, while the heterotic coupling is huge $g_H \sim l_s/l_H \sim 1/g_s$. Using heterotic – M-theory duality, one can find an alternative description in terms of M-theory compactified on the eleventh-dimensional interval $S^1/Z_2 \times T^4 \times T^2$. The M-theory length scale $l_M = g_H^{1/3} l_H$, so that $l_M^{-1} \sim 10^{14}$ GeV and the size of the eleventh dimension $R_{11} = g_H l_H \sim \text{TeV}^{-1}$. Thus, there are three large dimensions at the TeV (R_{11} and T^2) and four small dimensions of Planck length (corresponding to T^4) which invalidate the effective field theory description and makes this M-theory interpretation of no practical use.

3. LST and D-branes in LST

In this section we recall some aspects of LST and its weakly coupled version (DSLST). Using the idea of holography we review a dual description of the theory as a "non-critical" string, which allows to compute the spectrum and couplings in some appropriate regime. Moreover, this description provides a geometric set up which has some analogy with the scenario of [13]. We shall emphasize the similarities in this section, and we will make some remarks at the end of the paper. Finally, we also consider a different family of theories, on D-branes in LST, which shares some of the properties of LST.

3.1. Little String Theory

For simplicity, we first consider the six-dimensional LST (for a review, see [12] and references therein). One way to define this theory is the following. We start with a stack of k NS fivebranes in type II string theory with a string coupling g_s, and take the limit $g_s \to 0$. In this limit bulk degrees of freedom, including gravity, are decoupled, and one is left with a six-dimensional theory of strings without gravity. An alternative description is to consider the $g_s \to 0$ limit of type II on a singular K3 manifold. The two definitions are related by a T-duality.

This LST has the following properties [12] (in the conventions of the first definition):

(i) It has a unique scale M_s – the string mass scale of the original type II string theory.

(ii) In type IIA, the low energy theory is an $N = (2,0)$ six-dimensional SCFT, while in type IIB it is an $N = (1,1)$, $SU(k)$ gauge theory with a gauge coupling $g_{YM} \sim 1/M_s$.

(iii) It has a Hagedorn density of states and the Hagedorn temperature is $T_H = M_s/(2\pi\sqrt{k})$.

(iv) It is argued [16] that this theory is "holographically" dual to a higher dimensional string theory (with gravity): the type II string on

$$\mathcal{M} = R^{5,1} \times R_\phi \times SU(2)_k \ . \tag{3.1}$$

Here $R^{5,1}$ is the 5+1 dimensional Minkowski space in the directions of the worldvolume of the fivebranes. R_ϕ is the real line parameterized by a scalar ϕ, with a linear dilaton

$$\Phi = -\sqrt{\frac{1}{2k}}\phi \ . \tag{3.2}$$

ϕ is related to the radial direction r of the R^4 space transverse to the fivebranes: $\phi \sim \log r$. The $SU(2)_k$ is a level k WZW SCFT on the $SU(2) \simeq S^3$ background, where this three sphere is related to the angular coordinates of the transverse R^4. This background is obtained in the near horizon limit of the fivebranes, and describes the SCFT on the infinite "throat" (see figure 1(a)). The fivebranes might be thought of as sitting deep down the throat, namely, at $\phi \to -\infty$ ($r \to 0$) where the theory is strongly coupled ($\exp(2\Phi)$ is large (3.2)). On the other hand, as $\phi \to \infty$ ($r \to \infty$, towards the decoupled asymptotically flat space far from the fivebranes) the theory is weakly coupled.

(v) Off-shell observables in LST correspond to on-shell observables in string theory on $\mathcal{M}$ (3.1). Observables in the theory correspond to non-normalizable vertex operators,

namely, those whose wave function is exponentially supported at the weak coupling regime $\phi \to \infty$. There are also δ function normalizable operators whose role in the theory is less clear. The latter form a continuum of states, whose contribution to the density of states is a small fraction.

The holographic description above is useful to identify observables and their properties under the symmetries of the theory. However, correlation functions cannot be computed in perturbation theory because they are sensitive to the strong coupling regime down the throat. To resolve this strong coupling problem we shall "chop" the strong coupling regime of the throat (see figure 1).

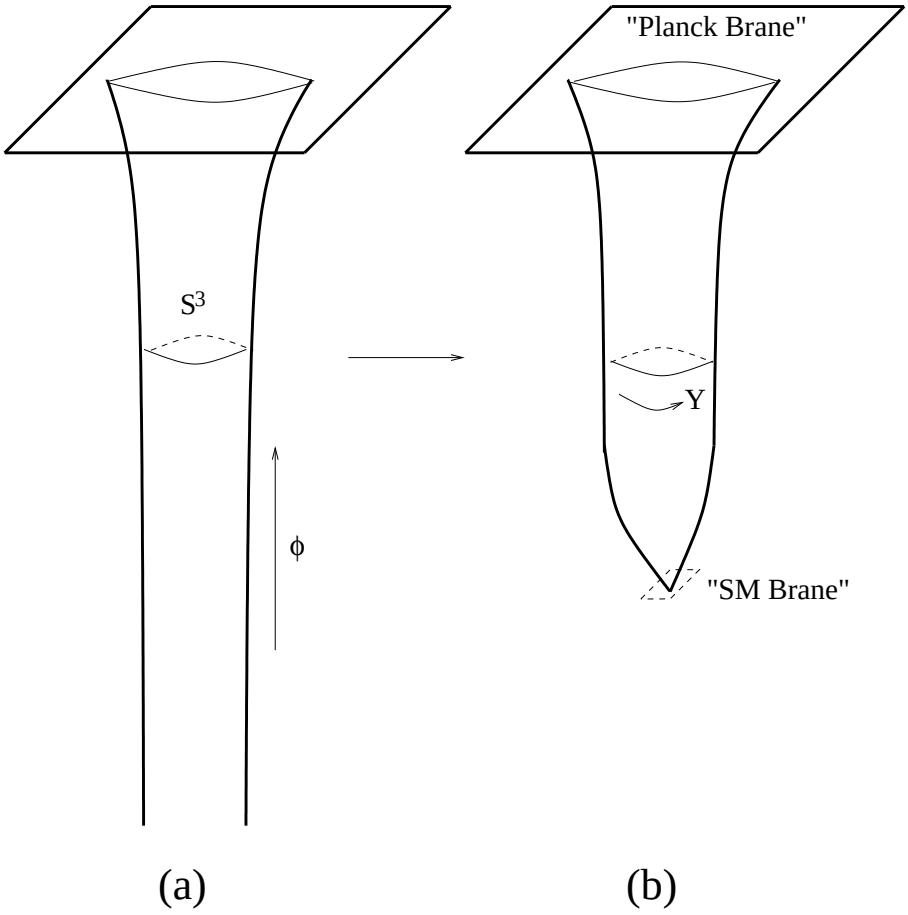

Fig. 1: *(a) The infinite throat background dual to strongly coupled LST; (b) The strong coupling region is chopped into a cigar-like geometry, whose tip is associated with a SM brane while the asymptotic region is associated with a Planck brane.*

It is convenient to decompose the $SU(2)_k$ SCFT on $S^1_Y \times SU(2)_k/U(1)$, where S^1_Y

is the Cartan sub-algebra of $SU(2)$, parameterized by a scalar Y, and the $SU(2)_k/U(1)$ quotient SCFT is equivalent to a level k, $N = 2$ minimal model. Then the throat SCFT becomes the product of an infinite cylinder $R_\phi \times S^1_Y$ (with a linear dilaton) times an $N = 2$ minimal model:

$$R_\phi \times S^3 \simeq R_\phi \times S^1_Y \times SU(2)_k/U(1) \ . \tag{3.3}$$

One way [17] to chop the strong coupling regime of the throat is to replace the infinite cylinder $R_\phi \times S^1_Y$ with the semi-infinite cigar [18] SCFT $SL(2)_k/U(1)$ (see figure 1(b)):

$$\mathcal{M} = R^{5,1} \times R_\phi \times S^1_Y \times \frac{SU(2)_k}{U(1)} \to R^{5,1} \times \frac{SL(2)_k}{U(1)} \times \frac{SU(2)_k}{U(1)} \ . \tag{3.4}$$

This corresponds to separating the k NS fivebranes on a transverse circle of radius L in the double scaling limit $g_s, L \to 0$ such that g_s/L is held fixed [19,17]. The string coupling takes its maximal value at the tip of the cigar where

$$g_s(tip) \equiv g_{lst} \sim \frac{g_s}{LM_s} \ , \tag{3.5}$$

while it approaches 0 as one goes away from the tip ($\phi \to \infty$) along the radial direction ϕ of the cigar. The scalar Y parameterizes the angular direction of the cigar whose radius is $R_{cigar} \sim \sqrt{k}/M_s$ asymptotically. The separation of the fivebranes introduces another scale in the theory. In type IIB it is the mass of a gauge boson corresponding to a D-string stretched between two NS5-branes, giving rise to a charged particle in the low energy $SU(k)$ gauge theory with mass

$$M_W^{IIB} \sim T_{D1} L = \frac{M_s^2 L}{g_s} = \frac{M_s}{g_{lst}} \ , \tag{3.6}$$

where $T_{D1} = M_s^2/g_s$ is the D-string tension. One may regard the above as chopping the infinite throat by SM branes (separated on a circle) near the tip of the cigar (see figure 1(b)).

So far we have considered the theory decoupled from gravity. The decoupling limit corresponds, in particular, to the limit $M_P \sim M_s/g_s \to \infty$. To keep gravity at the finite (although large) scale M_P observed in nature, we should relax the limit $g_s \to 0$ although the string coupling is still very small, as discussed in section 2. One may regard this as chopping the weak coupling regime of the semi-infinite cigar – the other side of the

original throat – by a Planck brane [4] (see figure 1(b)). We shall work in a scenario where $M_s, M_W^{IIB} \ll M_P$ and, therefore, the effects of gravity for $E \sim M_s$ are negligible.

This Double Scaled LST (DSLST) [17] has a weak coupling expansion parameter $g_{lst} = M_s/M_W^{IIB}$ when $M_W^{IIB} > M_s$ (we may however keep $M_W^{IIB} \ll M_P$). This allows one, in principle, to compute correlation functions perturbatively for processes at energies even larger than M_s (as long as they are sufficiently lower than M_W^{IIB}). On-shell correlators in the string theory (3.4) correspond, via holography, to off-shell Green's functions in the six-dimensional spacetime theory. From the analytic structure of the two point functions one can read the physical spectrum while the three point functions give rise to the couplings of physical states, via the LSZ reduction. The two and three point functions were computed in [17,21], with the following results:

1. *2-p-f:* The two point functions have a series of single poles, from which one can read the mass spectrum, followed by a branch cut (the poles correspond to the principal discrete series in the unitarity range of the $SL(2)/U(1)$ SCFT while the branch cut is due to the principal continuous series). The massless states correspond to photon multiplets in the low energy theory. They are followed by a discrete spectrum organized into Regge trajectories due to string excitations. The interpretation of the continuum is less clear, and is probably associated with "long strings" (see [22,23] and references therein). When g_s is finite the continuum in the spectrum is discretized.
2. *3-p-f:* The three point couplings allow, in particular, the decay of a massive discrete state into two massless states. Hence, one expects the stringy states to affect the form factor of the "photon" at energies of the order M_s.

Four-dimensional theories (at low energy) can be constructed in various ways, for instance:

(i) By compactifying two directions longitudinal to the fivebranes on a two torus, as described in section 2. The theory at energies below M_s and the compactification scale is an $N = 4$, $SU(k)$, four-dimensional gauge theory.

(ii) Four-dimensional LST whose low energy limit is an $N = 2$ SCFT in the moduli space of pure $N = 2$, $SU(n)$ gauge theory can be studied by considering the near horizon of a fivebrane wrapping a (singular) Riemann surface. Its holographic dual is [24] a type II string on

$$\mathcal{M} = R^{3,1} \times \frac{SL(2)_k}{U(1)} \times \frac{SU(2)_n}{U(1)} \ , \tag{3.7}$$

[4] More precisely, it is done by keeping the asymptotically flat regime "glued" to the throat, with the four-dimensional space transverse to the NS5-branes being compactified on T^4/Z_2 (similar to [20]) and with the appropriate number of NS5-branes as required by global issues.

with $k = \frac{2n}{n+2}$.

(iii) Richer four-dimensional LSTs can be obtained by replacing the level n, $N = 2$ minimal model in (3.7) with a richer $N = 2$ SCFT. For instance, replacing $\frac{SU(2)_n}{U(1)}$ by $[\frac{SU(2)_n}{U(1)} \times \frac{SU(2)_n}{U(1)}]/Z_n$, with $k = n/2$, leads [25] to an $N = 2$ SCFT with quark flavors.

(iv) Theories with $N = 1$ supersymmetry can be obtained by variations of the theories above, for instance, by orbifolding and/or by considering the decoupled theory on fivebranes in the heterotic string [26].

3.2. Theories on D-branes in LST

In this subsection we discuss the 3+1 dimensional theory on D4-branes stretched between NS5-branes when the string scale M_s is set, say, around 1 TeV. In particular, in such theories we will be able to discuss the spectrum and couplings of charged particles in theories with TeV strings without gravity.

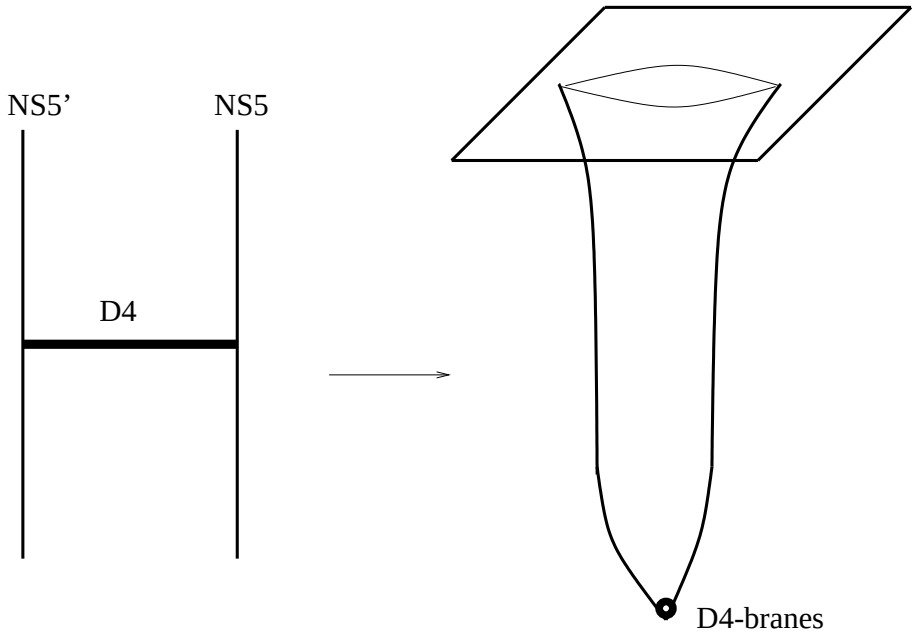

Fig. 2: *The decoupled theory on D4-branes stretched between NS5-branes is dual to D-branes near the tip of the cigar background.*

Consider the brane configuration in figure 2 (for a review, see [27]). An NS fivebrane is separated a distance ℓ from a (possibly differently oriented) NS' fivebrane, and N_c D4-branes are stretched between them. The low energy theory on the $R^{3,1}$ directions common

to all the branes is an $SU(N_c)$ gauge theory. The amount of supersymmetry of the theory depends on the relative orientation of the fivebranes. For instance, if the fivebranes are parallel the four-dimensional theory is $N = 2$ supersymmetric; a certain relative rotation of the fivebranes breaks it to $N = 1$ [28]. The (classical) YM coupling is

$$g_{YM}^2 = \frac{g_s l_s}{\ell} = g_{lst}(\ell) \ . \tag{3.8}$$

Following the discussion of the previous subsection, as $g_s \to 0$ (as well as $\ell/l_s \to 0$ such that $g_{YM}^2 = g_{lst}$ is held fixed), the decoupled theory is dual to a theory of D-branes near the tip of a cigar geometry (see figure 2) with $g_s(tip) \equiv g_{lst}(\ell)$ given in (3.8). At energies of the order M_s the spectrum becomes similar to that of LST: in the $N = 2$ case it is like a six-dimensional LST on $R^{5,1} \times SL(2)_2/U(1)$, while in the $N = 1$ case it is similar to a four-dimensional LST on $R^{3,1} \times SL(2)_1/U(1)$.

We can add matter to the theory in several ways. One way is to add D6-branes to the configuration and another is to add D4-branes on the other sides of the NS5-branes. In the second case the group that lives on the D4-branes in the center is called the "color group," whereas the one on the D4-branes sticking to the right (and/or to the left) is called the "flavor group." If the number N_c of colors equals (up to a model dependent numerical factor) the number of flavors N_f, the four-dimensional theory is conformal (in which case (3.8) is the exact gauge coupling). Moreover, we compactify the space transverse to $R^{3,1}$ on a six-dimensional space with volume V_6. [5] The four-dimensional Newton's constant is

$$\frac{1}{G_4} \sim M_P^2 \sim \frac{V_6}{g_s^2 l_s^8} \ . \tag{3.9}$$

Consider turning one of the NS fivebranes in the configuration above into a stack of fivebranes. Separating these fivebranes a distance L (say, on a circle transverse to the D4-branes) corresponds in the low energy four-dimensional SYM theory to changing certain parameters in the superpotential [28,29]. Such configurations allow to consider the physics of color-flavor open strings which are bound to the stack of NS5-branes perturbatively in $g_{lst}(L) = g_s l_s/L$ (3.5). Such open strings in the background of NS5-branes were studied in [30]. In particular, observables corresponding to "quarks" ((N_c, N_f) multiplets) and their excitations were identified and their two point functions were computed using the idea of

[5] Once this is done one should take care of global issues by allowing the appropriate total number of NS5-branes, introduce orientifolds, anti-D-branes, etc. (see comments below); we assume that this is done.

holography, following [17,21]. The spectrum of charged particles in such four-dimensional theories is thus very similar to the spectrum discussed in the previous subsection: the massless particles correspond to quarks, followed by a discrete Hagedorn spectrum with masses of the order M_s, and a continuum. Following the disk computations in [30] for the three point functions one expects, in particular, that a massive color-color string excitation can decay into a pair of quark-anti-quark.

Comments:

1. The configurations discussed above are only part of a globaly consistent brane configuration which includes more fivebranes, orientifolds, D-branes and anti-D-branes. All these extra objects can be located a distance of the order l_s away from the configurations above, hence physics due to the additional structure will show up at energies above the string scale.
2. Some systems of D4 and anti-D4-branes stretched between non-parallel NS5-branes are expected to be stable non-BPS brane configurations [31]. One might expect the supersymmetry breaking scale M_{SUSY} in such theories to be of the order of the string scale: $M_{SUSY} \sim M_s$.
3. The system discussed in this subsection is related to systems discussed in the previous subsection: there is a U-duality relating D4-branes intersecting NS5-branes to NS5-branes wrapped on a Riemann surface, which is holographically dual to the string theory on (generalizations of the) backgrounds of the form (3.7).

4. Phenomenology of TeV LST

Here, we discuss the main phenomenological implications of the above theories when the string tension and compactification scales are in the TeV region. Because of the tiny value of the string coupling $g_s \sim M_s/M_P \simeq 10^{-16}$, for all low energy consequences we can take the limit $g_s \to 0$, in which case gravity decouples and one is left over with LST having two sectors. The non-abelian (Standard Model) particles confined on NS5-branes (described by D-branes stretched between the NS5-branes) or on D-branes stretched between NS5-branes (described by open strings ending on the D-branes), and a sector of gauge singlet closed little strings trapped in the vicinity of the NS5-branes. Thus, there are three types of possible excitations revealing new physics:

(1) KK and winding modes of ordinary particles, signaling new dimensions at a TeV.

(2) String oscillations of the quarks, leptons and familiar gauge bosons.

(3) Vibrational excitations of the little string at the TeV scale. These are unique to this framework.

There are three types of theories, each with different signatures: type IIB, type IIA and theories with D-branes stretched between NS5-branes. We will consider them in turn.

Type IIB models:

In these the Standard Model gauge interactions are described by a six-dimensional theory of 0-branes obtained as endpoints of D-strings on the NS5-branes. Their tension T_0 is determined by eq. (3.6) and can be identified with the mass of the W-boson, $T_0 \equiv T_{D1}L \simeq M_W$. [6] The four-dimensional gauge coupling (2.2) is determined by the area of the two-dimensional compact space along the NS5-branes, which implies that the compactification scale is an order of magnitude lower than the string scale:

$$M_c \equiv \frac{1}{R_5} \sim \frac{1}{R_4} = g_{YM} M_s \,. \tag{4.1}$$

It follows that the first effects of charged particles beyond the Standard Model that would be encountered in particle accelerators are due to the production of KK excitations in the two extra dimensions [32]. Neutral states will also appear at the TeV range; we shall discuss them below.

Type IIA models:

In these the gauge degrees of freedom are described by strings, obtained as endlines of D2-branes on the NS5-branes. Their tension $T_1 \equiv T_{D2}L$ can be obtained by T-duality from type IIB along, say, the direction X^4, so that the gauge coupling (2.2) is given by the ratio of the two radii. As a result, $T_1 = T_0/R_4 \equiv M_W/R_4$ leading to

$$T_1 = \frac{M_W}{R_4} = \frac{M_W M_c}{g_{YM}^2} \,. \tag{4.2}$$

On the other hand, $T_1 = T_{D2}L = M_s^3 L/g_s$, which combined with eq. (3.5) yields:

$$\text{type IIA}: \qquad g_{lst} = \frac{M_s^2}{T_1} \,. \tag{4.3}$$

[6] In theories with large supersymmetry there are also magnetically charged particles, however, those can be pushed above the $M_s \sim$ TeV scale. More realistic models where supersymmetry is broken down to $N = 1$ or $N = 0$ can be obtained, say, by appropriate orbifoldings; in such theories magnetically charged particles are projected out.

From eq. (4.2), M_W is identified with the dual compactification scale along the X^4 direction with respect to the charged string tension T_1. If the fundamental string tension M_s^2 is lighter than T_1, then $M_s^2 R_4 < T_1 R_4 = M_W$ and closed little strings have windings at energies lower than M_W, which is excluded experimentally. Thus, the tension T_1 of the charged gauge states is less than that of the little strings, $\sqrt{T_1} < M_s$, and in the energy interval between the two tensions we will have an effective superconformal theory of tensionless strings (see also [33]). However, from eq. (4.3), $g_{lst} > 1$ and we cannot reliably compute in little string perturbation theory. It follows that in this case the first effects of charged particles that would be encountered in particle accelerators are KK modes of one dimension along the X^5 direction and/or charged string excitations with tension T_1, depending whether $M_c \equiv R_5^{-1}$ is less or bigger than M_W/g_{YM}^2. In both the type IIA and IIB frameworks the coupling of the little strings to the standard model matter is unknown.

D-branes in LST:

In type II models where gauge interactions emerge from D-branes stretched between NS5-branes, the low energy physics is described by the theories on D-branes in LST, discussed in section 3.2. Note that the mass of W bosons corresponds now to the separation of the D-branes and is independent of separations of NS5-branes.

In this case, Standard Model particles have charged excitations due to windings of open strings in the directions transverse to the D4-branes but along the two extra compact dimensions of the NS5-branes. The energy of these excitations is $M_s^2 R_c$, where R_c is the compactification scale. If $R_c < l_s$ we can T-dualize $R_c \to \tilde{R}_c = l_s^2/R_c > l_s$. In this case the D4-branes turn into D5-branes wraped on the compact direction $\tilde{R}_c$, and $g_{lst}(\ell)$ in eq. (3.8) turns into $\tilde{g}_{lst}(\ell) = g_{lst}(\ell) l_s/R_c$. Charged excitations in this direction correspond now to KK modes of open strings which are somewhat lighter than the string scale. In fact, the weak coupling condition $\tilde{g}_{lst} < 1$ gives $l_s > R_c > g_{lst} l_s = g_{\rm YM}^2 l_s$. Thus, both the compactification and the string scales are in the TeV region and in all these cases the energy of such charged particles is around the TeV scale while little strings are weakly coupled.

There are also KK modes of open strings in the direction along which the D4-branes are stretched as well as windings along the directions transverse to both the D4 and the NS5-branes; those are very weakly coupled (and decouple in the $g_s \to 0$ limit). In addition, there are of course fundamental open string oscillator modes that are also charged under

SM gauge interactions and have TeV masses.

The common thread of all three cases is the existence of a neutral sector described by closed fundamental little strings that survive in the limit $g_s \to 0$. They have non-trivial interactions among themselves with a coupling g_{lst} given in eq. (3.5). Perturbative computations can therefore be trusted when $g_{lst} < 1$. Using eqs. (4.3), (4.2) and (3.6), one obtains:

$$\begin{aligned} \text{type IIA}: &\qquad g_{lst} = \frac{M_s^2}{T_1} = g_{YM}^2 \frac{M_s^2}{M_W M_c} \ , \\ \text{type IIB}: &\qquad g_{lst} = \frac{M_s}{M_W} \ , \\ \text{D}-\text{branes in LST}: &\qquad g_{lst} = g_{YM}^2 = \frac{g_s}{\ell M_s} \ . \end{aligned} \tag{4.4}$$

Recall that for theories of D-branes in the presence of NS5-branes, g_{lst} is independent of the W boson mass which is determined by the separation of the D-branes and not of the NS5-branes.

It follows that the discussion of the perturbative spectrum of section 3 is strictly speaking valid for the theories of D-branes in LST if the separation of NS5-branes is larger than $g_s l_s \sim 1/M_P$. The little string excitations can be produced in particle accelerators if they dispose sufficient energy, or they can lead to indirect effects in various processes, as the effects of TeV string models based on type I theory [34].

The perturbative spectrum occurs at [17,21]

$$M_{n,m}^2 = \frac{2M_s^2}{k}(n-1)(2m-n) \ , \qquad 2m+1 > 2n > 2m+1-k \ , \qquad n \in Z \ . \tag{4.5}$$

The pole at $n = 1$ corresponds to the light SM particle. The other poles at $M_{n,m}^2 \sim M_s^2/k$ are KK-type excitations, due to the asymptotic radius of the cigar. Each set of poles on the up-side-down parabola (4.5) is followed by a branch cut starting at the maximum of the parabola, that we discussed in the previous section.

Similarly, observables corresponding to string excitations N create from the vacuum particles with masses

$$M_{n,m;N}^2 = M_{n,m}^2 + N M_s^2 \ . \tag{4.6}$$

Hence, each of the particles in (4.5) is followed by a Regge trajectory of string excitations [7].

[7] For charged open little strings (open strings in the background of NS5-branes), some factors of 2 should be added in eqs. (4.5), (4.6) relative to the neutral closed little strings sector; see for instance eqs. (B.14), (B.15) in [30].

It is interesting to consider the thermodynamics of LST at a TeV; this can be done using its holographic description [35,36,37,38,39]. Strongly coupled LST has a Hagedorn density of states and the Hagedorn temperature is $T_H = M_s/(2\pi\sqrt{k})$ [35]. At high energy the entropy is

$$S = \beta_H E + \alpha \log E + O(1/E) \ , \tag{4.7}$$

leading to the temperature-energy relation

$$\beta = \frac{\partial S}{\partial E} = \beta_H + \alpha/E + O(1/E^2) \ . \tag{4.8}$$

The sign of α indicates if $T_H = 1/\beta_H$ is a limiting temperature (where the energy density diverges as T approaches T_H from below) or a temperature where a phase transition might occur. Recently, α was computed and was shown to be negative [39]. This suggests that a phase transition is expected at $T \sim M_s$, similar to QCD. The nature of the high temperature behavior of the theory might have some interesting consequences in the physics of the early universe.

Weakly coupled LST ($g_{lst} < 1$) has of course the same high energy thermodynamics (when $E \gg M_s/g_{lst}$). However, at intermediate energies $M_s < E < M_s/g_{lst}$ the weakly coupled little string excitations possess a Hagedorn density of states with $T_H = M_s/(2\pi)$ [17].

Finally, we remark that gauge coupling Unification, the one concrete quantitative success of the supersymmetric standard model [2], can be accommodated in a way parallel to [40]. There it was shown that under general conditions, placing the color and weak interactions on two different sets of branes (extended in different directions in the internal compact space) implies one relation among the three gauge couplings which naturally leads to the correct value of the weak mixing angle, provided that we choose the fundamental scale to be at a few TeV.

5. The Hierarchy Problem

In the framework we described here, the hierarchy between the Planck and the string scales is attributed to the smallness of the string coupling. A small string coupling is rather natural when supersymmetry is broken due to the runaway potential generated for the dilaton [41]. In a usual scenario where the YM couplings are of the order g_s such a runaway behavior is a serious problem. On the other hand, in the LST scenario

considered in this note, g_{YM} is determined by geometrical data, while g_s is an independent parameter. Yet, although $g_s \ll 1$ is required to set the observed Newton's constant, it should not runaway all the way to 0. In this section, we describe a mechanism determining the expectation value (VEV) of the dilaton and the conditions for generating the desired hierarchy.

Let us first remark that if the value of the string coupling is chosen to be very small by hand, the resulting hierarchy is obviously stable under radiative corrections even around a non-supersymmetric string vacuum. This should be contrasted with the large dimension framework [6] where stability of hierarchy requires that massless bulk fields propagate in more than one large compact dimensions [42]. Moreover, the vacuum energy in a non-supersymmetric vacuum of the theories we described here behaves at most as $M_s^4 \sim (\mathrm{TeV})^4$ (see also [43] and the discussion below). This should be again contrasted with the framework of large dimensions which suffers in general from the usual quadratic divergences $\sim M_s^2 M_P^2$, unless the bulk is supersymmetric [6,44].

Dynamically determining the dilaton by minimizing an effective potential faces the following problem. Since the dilaton plays the role of string loop expansion parameter, a generic non-trivial potential would mix several orders of perturbation theory (as well as eventually non-perturbative effects) and, in general, the minimum would be at a point where different powers of g_s compete and, as a result, perturbation theory is unreliable. Moreover, the value of the coupling is in general expected to be of order unity. A possible exception using non-perturbative contributions such as several condensates [45] appears very unnatural in our case, since non-perturbative factors are extremely suppressed in the desired very weak coupling limit.

One way to evade this problem is through the appearance of logarithms. These can arise from loops of particles having gauge interactions with masses depending on the string coupling. The first difficulty is that gauge theories on NS5-branes are independent of the string coupling. One should therefore introduce a new gauge (hidden) sector living on D-branes and thus having a gauge coupling given by $g_s^{1/2}$. The second difficulty is that massive particles on D-branes have in general masses set by their separation, their motion or the string scale itself (for string excitations) all of which are independent of g_s. In these cases, loop effects cannot produce logarithms of g_s. However when masses are induced radiatively, they depend on the string coupling and can give rise to logs. One such example arises when there is an anomalous $U(1)$. The anomaly is cancelled by an appropriate shift of an axion from the Ramond-Ramond sector and the abelian gauge field acquires a mass

$m_A = g_s^{1/2} M_s$. Integrating out this field, one obtains a potential term proportional to $m_A^4 \ln m_A$, or equivalently (in the string frame):

$$V_{\rm eff} = g_s^2 (v_1 \ln g_s + v_2) M_s^4 + c M_s^4 \,, \tag{5.1}$$

with $v_{1,2}$ and c numerical constants. The first two terms proportional to $v_{1,2}$ correspond to two string loops contributions (genus 2), while c arises at the one loop (genus 1).

The effective potential (5.1) has an extremum at

$$\langle g_s \rangle = e^{-1/2 - v_2/v_1} \,, \tag{5.2}$$

which is a minimum when v_1 is positive. This minimum can be exponentially small when v_2 is just one or two orders of magnitude bigger than v_1, which is not unreasonable since v_1 is determined entirely from the loop of the anomalous $U(1)$ while v_2 receives contributions from all string modes.

Note that in general, in the presence of D-branes, one may expect an additional contribution to $V_{\rm eff}$ proportional to g_s, arising from genus 3/2. Such a term would destabilize the minimum (5.2) and is assumed to vanish. In fact this condition is related to the problem of fine tuning the cosmological constant. In the above example (5.1), we should therefore impose

$$c = -\langle g_s \rangle^2 (v_1 \ln \langle g_s \rangle + v_2) + \mathcal{O}(\langle g_s \rangle^4) \,. \tag{5.3}$$

Another example of logarithmic corrections to the potential may be provided in models of the Coleman-Weinberg type, where a classically massless scalar field with a tree-level potential acquires a non-trivial VEV driven by a negative squared mass generated radiatively. In this case, the scalar potential takes the form $V_{\rm eff} \sim \Phi^4/g_s - \mu^2 \Phi^2$, up to $\ln \Phi$ corrections in both terms. Then, its minimization fixes $\langle \Phi \rangle \propto g_s^{1/2}$ and leads to $V_{\rm eff} \sim g_s \ln g_s$ which is similar to the expression (5.1) that we studied above.

An alternative possibility of fixing the dilaton without generating a potential would be during the cosmological evolution of the universe, following the suggestion of Damour and Polyakov [14]. The basic requirement is that all couplings and masses of the effective theory should depend on the dilaton through the same function. If in addition this function has an extremum, the cosmological evolution will "push" the dilaton towards this extremum. This happens during matter dominated era, crossing mass thresholds in radiation dominated, as well as during any period of inflation. As a result, the dilaton couples quadratically to matter and its mass can vanish without causing any dangerous long range force.

The main requirement of a universal functional dependence seems however very unlikely to be satisfied in heterotic and type I string vacua. On the contrary, theories on NS5-branes seem to provide a natural framework for realizing such a requirement, since in the string frame the matter action is independent of the dilaton while graviton kinetic terms may acquire a non-trivial dilaton dependence:

$$\mathcal{L} = F(g_s)R + \mathcal{L}_{\rm matter}\,, \tag{5.4}$$

where $F(g_s) = 1/g_s^2 +$ higher order and non-perturbative corrections. It follows that upon rescaling the metric into the Einstein frame, all mass parameters of the matter Lagrangian will depend on the single "universal" function F. It is not however clear under what conditions F would have an extremum at a tiny value of g_s.

The dilaton may approach but not precisely reach the extremum of F in cosmological time [46]. This results in a small universal linear coupling of the dilaton to matter – but not to radiation – proportional to the fractional deviation α of the dilaton's present position away from its minimum. Such a scalar admixture to gravity has several possible observational consequences, including the bending of light and the Shapiro time delay of signals [46,47]. The most stringent bounds come from primordial nucleosynthesis, and they constrain the present value of α to be less than a few percent [47]. A precision test, possibly improving the present limit of α by over an order of magnitude, will take place in the relativistic gyroscope (or Gravity Probe B) experiment that will be launched in 2002.

In general, small flavor-dependent effects are expected to spoil the exact universality of the coupling of the dilaton through the function F. In the string frame these are expected to show up as small g_s-dependent corrections to the various gauge-invariant terms in the matter Lagrangian of eq. (5.4). This results in a linear coupling of the dilaton to matter which is flavor-dependent and, therefore, leads to violations of the principle of equivalence – estimated to be proportional to the product of $\alpha \times g_s$. These are potentially observable in the upcoming satellite experiment STEP, which will test the principle of equivalence to one part in 10^{18} [14]. Since g_s grows with M_s and V_6, the predicted violations are larger for string scale above a TeV or bulk volume above a TeV^{-6} (see footnote 2).

6. Remarks

We end with some comments, first on the cosmological constant. In models with infinitesimal string coupling, it seems that the vacuum energy may be consistent with the present experimental bound if the perturbative contributions are arranged to vanish in one and two loops, while non-perturbative corrections appear to be extremely suppressed. Indeed, the three loop contribution is of order $g_s^4 M_s^4 \sim M_s^8/M_P^4$ which is just of the order suggested by present observations for $M_s \sim 1$ TeV. This may provide a new framework for explaining the smallness of the cosmological constant which deserves further investigation.

The theory of NS fivebranes with the string scale set equal to the electroweak scale and with a very small asymptotic string coupling realizes several recent ideas in explicit string theory backgrounds.

For example, the tip of the cigar is a concrete realization in string theory of what one would call in [13] "the d-dimensional negative tension brane" (see figure 1). Unlike possible realizations of warped compactification scenaria in string theory, here the theory at high energies is *not* a CFT; it is a string theory with a scale M_s coupled (weakly) to gravity. Nevertheless, there is an analogy between the dilaton and the y coordinate responsible for the exponential hierarchy in warped compactification scenaria.

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Inflation, Deflation, and Frame-Independence in String Cosmology

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Abstract

The inflationary scenarios suggested by the duality properties of string cosmology in the Brans-Dicke (or String) frame are shown to correspond to accelerated contraction (deflation) when Weyl-transformed to the Einstein frame. We point out that the basic virtues of inflation (solving the flatness and horizon problems, amplifying vacuum fluctuations, etc.) have physically equivalent counterparts in the deflationary (Einstein-frame) picture. This could be the answer to some objections recently raised to superstring cosmology.

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1. Introduction

A potential source of difficulty for extended-inflation models [1] based on a Brans-Dicke theory of gravity [2] is the choice of the correct frame (metric) in which to describe the space-time geometry at a cosmological level. One may wonder, in particular, in which frame the metric should be of the inflationary type, and satisfy the conditions required to avoid the problems of the standard cosmological scenario.

While the choice of the Einstein (E) frame (in which the Einstein-Hilbert term takes the General-Relativity form) usually simplifies calculations and is quite popular, there are physical motivations for choosing instead the Brans-Dicke (BD) frame, in which matter couples to the metric-tensor in the standard way [3]. Arguments in favour of the BD choice can also be given in string theory [4], where the BD frame metric coincides with the σ-model metric to which test strings are directly coupled. Thus free string motions follow geodesic surfaces with respect to the BD (not the E) metric.

The physical observable properties of a given model should be independent, of course, from the field redefinition (Weyl rescaling) connecting BD and E frames. And indeed, in the case of extended inflation, the metric describing a phase of power-law inflation (with variable Newton constant) in the BD frame, is transformed into a metric, which is still describing power inflation (of the slow-roll type, with exponential potential) in the E frame, as discussed for instance in [5].

In a string theory context, the role of the BD scalar is played by the dilaton field. In such case, as pointed out in [6], there appear to be serious difficulties in arranging a successful phase of dilaton-driven, power-law, extended inflation, at least if theoretically motivated dilaton potentials are used. On the other hand, the cosmological equations obtained from the low-energy string effective action show that the dilaton can drive (even in the absence of a potential) a phase of accelerated expansion. This phase, supposedly describing the Universe before the big-bang (so-called "pre-big-bang" [7]), is characterized by being just the "dual" counterpart (in the sense of ref. [8]) of the "post-big-bang" standard cosmology.

The "pre-big-bang" phase corresponds, in the BD frame, to a superinflationary expansion. When transformed to the E frame, however, the same metric describes, as we shall see, a contracting Universe. Apparently, this represents a difficulty for the whole scenario, since the presence or absence of inflation (and of its bonuses) would seem to become frame-dependent.

In this paper we shall show that, on the contrary, even in the E frame the

solutions of the string cosmology equations provide an adequate description of the inflationary phase, provided we generically mean, by "inflation", a phase of cosmological evolution that is able to avoid the problems (see for instance [9]) related to the decelerated kinematics of the standard cosmological model.

At the same time, and irrespectively of strings and/or BD theory, we shall argue that the solution of many of the standard-cosmology problems achieved by inflation is also possible through the introduction of an early phase of accelerated contraction, that we shall call deflation. This will be the content of the following section.

2. Inflation vs. deflation

It is well known that there are three possible classes of inflationary evolution [10], corresponding to a curvature scale that is constant (De Sitter inflation), decreasing (power inflation) or increasing (superinflation). Less known, however, seems to be the fact that in a phase of growing curvature the solution of the standard cosmological puzzles can be realized in two ways, namely by a metric describing *either* accelerated expansion, $\dot{a} > 0, \ddot{a} > 0$, *or* accelerated contraction, $\dot{a} < 0, \ddot{a} < 0$ (a is the scale factor of a homogeneous and isotropic model, and a dot denotes differentiation with respect to cosmic time).

A possible equivalence of superinflation and accelerated contraction is clearly pointed out by an elementary analysis of the so-called flatness problem. If we want the contribution of the spatial curvature k to be suppressed with respect to the other terms of the cosmological equations, then the ratio

$$r_1 = \frac{k}{a^2 H^2} = \frac{k}{\dot{a}^2} \quad , \quad H \equiv \dot{a}/a \; , \tag{2.1}$$

must tend to zero during the inflationary era. Such a condition is clearly satisfied by a metric that behaves, for $t \to +\infty$, as

$$a \sim t^{\alpha} \quad , \quad t > 0 \quad , \quad \alpha > 1 \; , \tag{2.2}$$

but also by a metric, which, for $t \to 0_-$, behaves as

$$a \sim (-t)^{\beta} \quad , \quad t < 0 \quad , \quad \beta < 1 \; . \tag{2.3}$$

The case (2.2) corresponds to power inflation, and includes the standard De Sitter exponential inflation in the limit $\alpha \to \infty$. The second case, (2.3), corresponds, for $\beta < 0$, to the well-known case of pole inflation (superinflationary

expansion, $\dot{a}, \ddot{a}, \dot{H}$ all positive). For $0 < \beta < 1$ it describes instead an accelerated contraction, or deflation ($\dot{a}, \ddot{a}, \dot{H}$ all negative). In both cases the curvature scale is growing, and $H, \dot{H}$ diverge as $t \to 0_-$.

A deflationary phase (2.3), with $0 < \beta < 1$, may also provide a solution to the so-called horizon problem. The presently observed large-scale homogeneity and isotropy requires the proper size of the particle horizon to become large enough during the inflationary era, and to go to infinity in the limiting case in which inflation extends for ever in the past. This means that the integral

$$d_p(t) = a(t) \int_{t_1}^{t} dt' a^{-1}(t') \tag{2.4}$$

must diverge, if a is the inflationary scale factor, when t_1 approaches the maximal past extension of the cosmic time coordinate for the given cosmological manifold.

For the metric (2.3) such a limiting time is $-\infty$, and $d_p \to \infty$ for $t_1 \to -\infty$, so that there are no particle horizons in a phase of accelerated contraction.

As a consequence of accelerated contraction, causally connected regions are pushed out of the event horizon, just as in the standard inflationary expansion. It is true that the proper size of a causally connected region tends to contract, asymptotically, like the scale factor. For a patch of initial size $d_1 \sim (-t_1)$ one finds in fact, from eqs. (2.3) and (2.4), that $d_p \to [a(t)/a(t_1)]d_1$ for $|t| << |t_1|$. However, the proper size of the event horizon, defined by

$$d_e(t) = a(t) \int_{t}^{t_2} dt' a^{-1}(t') \tag{2.5}$$

(t_2 is the maximal allowed future extension of the cosmic time coordinate), contracts always faster than d_p. Indeed, $t_2 = 0$ for the metric (2.3), and one finds that $d_e(t) \sim (-t)$ for $t \to 0$. The ratio of the two proper sizes at small t

$$r_2(t) = \frac{d_p(t)}{d_e(t)} \sim (-t)^{\beta - 1} \tag{2.6}$$

shows that the causally connected regions will always cross the horizon, asymptotically, not only in the case of superinflationary expansion ($\beta < 0$), but even in the deflationary case ($0 < \beta < 1$).

We note, for later convenience, that the conditions for a successful resolution of the horizon and flatness problem, when expressed in terms of the conformal time coordinate η ($a = dt/d\eta$), are exactly the same for both superinflationary expansion and accelerated contraction. Moreover, if the contracting phase is long

enough to solve the horizon problem, then also the flatness problem is automatically solved (and vice versa), as in standard inflation.

Indeed the ratio r_2 scales in conformal time like η^{-1}, while the ratio r_1 scales like η^2. The horizon problem is solved if $r_2(\eta_f)$, evaluated at the end of the accelerated evolution ($\eta = \eta_f$), is larger than the present value $r_2(\eta_0) \simeq 1$, rescaled down at η_f. This implies

$$\frac{|\eta_i|}{|\eta_f|} \gtrsim \frac{|\eta_0|}{|\eta_f|} \simeq 10^2 (\frac{T_{rh}}{eV}) \, . \tag{2.7}$$

Here η_i denotes the beginning of the contracting (or expanding) accelerated evolution, T_{rh} the final reheating temperature at $\eta = \eta_f$, and the last equality holds in the hypothesis of standard, adiabatic, radiation-dominated and matter-dominated expansion from η_f down to the present time η_0.

The solution of the flatness problem, on the other hand, is obtained if the ratio r_1 at the end of the accelerated phase is tuned to a value that is small enough, so that the subsequent decelerated evolution leads to a present value of r_1 satisfying the condition $r_1(\eta_0) \lesssim 1$. This means

$$(\frac{\eta_f}{\eta_i})^2 \lesssim (\frac{\eta_f}{\eta_0})^2 \, , \tag{2.8}$$

which is clearly equivalent to eq. (2.7), and which implies a resolution of the flatness and horizon problems (as well as of their rephrasing in terms of the entropy [9]) for both expanding and contracting metrics of the type (2.3).

Besides solving the kinematical problems, a phase of successful inflation is also expected to efficiently amplify the vacuum fluctuations of the metric background. We shall conclude this section by noting that such an amplification can also be provided by a long period of deflation.

Consider, for instance, the amplification of tensor perturbations h^ν_μ (similar arguments hold for the scalar case also). In a four-dimensional conformally flat background, the wave equation for each Fourier component of h can be written in terms of the rescaled variable $\psi = ah$ as [11]

$$\psi'' + (k^2 - \frac{a''}{a})\psi = 0 \tag{2.9}$$

(a prime denotes differentiation with respect to conformal time). In a realistic case, the phase of accelerated evolution is followed by the standard radiation-dominated expansion, with $a \sim \eta$, and the amplification of the fluctuations can be

described as a process of graviton production from the vacuum (such an approach will be used in Section 3). Equivalently, in a Schrödinger-like language, the process corresponds to a parametric amplification of the perturbation wave function [11], which is oscillating at $\eta \to \pm\infty$, and evolves with a power-law behaviour in the regions where the co-moving frequency k is negligible with respect to the effective potential a''/a of eq. (2.9).

By inserting into (2.9) a generic parametrization (in conformal time) of the accelerated metric, $a(\eta) = (-\eta)^{-\delta}$, one finds indeed that the solution behaves like

$$h \sim A_{\pm} \frac{e^{\pm ik\eta}}{a} \quad , \quad k\eta >> 1 \tag{2.10}$$

$$h \sim A + B\frac{(-\eta)}{a^2} = A + B(-\eta)^{1+2\delta} \quad , \quad k\eta << 1 \tag{2.11}$$

($A_{\pm}, A, B$ are integration constants). In the case of accelerated expansion ($\delta > 0, a \to \infty$ for $\eta \to 0_-$), the perturbations are amplified because their amplitude tends to stay constant in the $\eta \to 0$ limit, instead of decreasing adiabatically as in the oscillating regime (2.10).

In the case of deflation ($\delta < 0, a \to 0$ for $\eta \to 0_-$), the amplification process is even more efficient than in the previous case, as the amplitude of h grows (with respect to the adiabatic red-shift of the subsequent radiation-dominated expansion) even in the oscillating regime. Moreover, as shown by eq. (2.11), h may even grow asymptotically (instead of being constant) provided $\delta < -1/2$. As we shall see in Section 3, this condition is satisfied in particular, in the E frame, by a 3-dimensional phase driven by stretched strings.

Note that the amplification coefficient corresponding to a phase of accelerated contraction is different, in general, from the one corresponding to a phase of accelerated expansion. It is just because of this difference that the perturbation spectrum may remain unchanged, when an inflationary background is transformed into a deflationary one through a conformal rescaling, as we shall see in the following Section.

3. Pre-big-bang cosmology in the Brans-Dicke and Einstein frames

In a string cosmology context [7,12], a global (at least semi-quantitative) description of the evolution and symmetries of the early Universe is expected to be provided by the low-energy string effective action, possibly supplemented by the action S_m for macroscopic matter sources:

$$S = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{|g|}e^{-\phi}[R + (\partial_\mu\phi)^2 - \frac{1}{12}H^2_{\mu\nu\alpha} + V] + S_m \tag{3.1}$$

Here $H_{\mu\nu\alpha}$ is the antisymmetric tensor field strength, and V a (possibly non-zero) dilaton potential.

In this paper we will consider a $(d+1)$-dimensional, anisotropic metric background of the Bianchi I type, with time-dependent dilaton,

$$g_{00} = 1 \quad , \quad g_{ij} = -a_i^2 \delta_{ij} \quad , \quad \phi = \phi(t) \quad , \quad i,j = 1,2,...,d \tag{3.2}$$

and with vanishing $H_{\mu\nu\alpha}$ and $V(\phi)$. The additional matter sources, which are decoupled from the dilaton in this frame, will be represented by a perfect fluid with anisotropic pressure:

$$T_0^0 = \rho \quad , \quad T_i^j = -p_i \delta_i^j = -\gamma_i \rho \delta_i^j \ . \tag{3.3}$$

By defining as usual [8,7,12]

$$\overline{\phi} = \phi - \ln \sqrt{|g|} \quad , \quad \overline{\rho} = \rho \sqrt{|g|} \quad , \quad \overline{p} = p \sqrt{|g|} \tag{3.4}$$

the field equations following from the variation of the action (3.1) can be written in the form [8]

$$\dot{\overline{\phi}}^2 - 2\ddot{\overline{\phi}} + \sum_i H_i^2 = 0 \tag{3.5}$$

$$\dot{\overline{\phi}}^2 - \sum_i H_i^2 = \overline{\rho} e^{\overline{\phi}} \tag{3.6}$$

$$2(\dot{H}_i - H_i \dot{\overline{\phi}}) = \overline{p}_i e^{\overline{\phi}} \tag{3.7}$$

where $H_i = \dot{a}_i / a_i$, and we use units in which $8\pi G = 1$. Their combination gives the usual conservation equation

$$\dot{\overline{\rho}} + \sum_i H_i \overline{p}_i = 0 \ . \tag{3.8}$$

By applying the general procedure illustrated in [7], the background field variables can be separated, and the equations can be integrated exactly, by introducing a suitable time-like coordinate x such that

$$\overline{\rho} = \frac{1}{L} \frac{dx}{dt} \tag{3.9}$$

(L is a constant with dimensions of length, in such a way that x is dimensionless). For constant γ_i we obtain the following general exact solution of eqs.(3.5–3.7) (a similar problem was first solved in a different context in [13]):

$$a_i = a_{0i} |(x - x_+)(x - x_-)|^{\gamma_i/\alpha} \left| \frac{x - x_+}{x - x_-} \right|^{\alpha_i} \tag{3.10}$$

$$e^{\overline{\phi}} = e^{\phi_0}|(x-x_+)(x-x_-)|^{-1/\alpha}|\frac{x-x_+}{x-x_-}|^{-\sigma} \tag{3.11}$$

$$\overline{\rho} = \frac{\alpha}{4L^2}e^{\phi_0}|(x-x_+)(x-x_-)|^{(\alpha-1)/\alpha}|\frac{x-x_+}{x-x_-}|^{-\sigma} \tag{3.12}$$

where

$$\alpha = 1 - \sum_i \gamma_i^2 \quad , \quad \sigma = \sum_i \alpha_i \gamma_i \quad , \quad \alpha_i = \frac{\alpha x_i + \gamma_i(\sum_i \gamma_i x_i - x_0)}{\alpha[(\sum_i \gamma_i x_i - x_0)^2 + \alpha(\sum_i x_i^2 - x_0^2)]^{1/2}}$$

$$x_\pm = \frac{1}{\alpha}\{\sum_i \gamma_i x_i - x_0 \pm [(\sum_i \gamma_i x_i - x_0)^2 + \alpha(\sum_i x_i^2 - x_0^2)]^{1/2}\} \tag{3.13}$$

and a_0, ϕ_0, x_0, x_i are integration constants.

This solution has various interesting properties, which we shall discuss elsewhere [14]. Here we only note that there are two curvature singularities at $x = x_\pm$, and that the region between the singularities is unphysical, in the sense that the critical density parameter

$$\Omega(x) \equiv \frac{\rho e^\phi}{(d-1)\sum_i H_i^2} = \frac{(x+x_0)^2 - \sum_i(\gamma_i x + x_i)^2}{(d-1)\sum_i(\gamma_i x + x_i)^2} \tag{3.14}$$

becomes negative. This parameter tends to zero at the singularities, and in this limit the metric (3.10) goes over to the vacuum solutions of string cosmology [15,8]. For $x \to x_\pm$ one finds indeed

$$a_i(t) \sim |t - t_\pm|^{\beta_i^\pm} \ , \tag{3.15}$$

where

$$\beta_i^\pm = \frac{x_i \pm \gamma_i x_\pm}{x_0 + x_\pm} \quad , \quad \sum_i (\beta_i^\pm)^2 = 1 \ . \tag{3.16}$$

However, because of the neglect in the original action (3.1) of truly "stringy" contributions (such as α' and loop corrections), this solution is not expected to provide a reliable description of the very high curvature regime. The appropriate range of validity of the solution is instead the large $|x|$ limit, and in particular $x \to -\infty$, where it provides a typical example of pre-big-bang evolution, characterized by acceleration and growing curvature scale [7].

If we consider, in particular, the isotropic case with negative pressure ($a_i = a, \gamma_i = \gamma < 0$ for all d spatial directions), then at large negative x we have $|x| \sim |t|^{\alpha/(2-\alpha)}$, and the solution (3.10-3.12) becomes, in this limit,

$$a(t) \sim (-t)^{2\gamma/(1+d\gamma^2)} \quad , \quad \overline{\phi}(t) \sim -\frac{1}{\gamma}\ln a$$

$$\phi = \overline{\phi} + d \ln a \sim \frac{d\gamma - 1}{\gamma} \ln a \quad , \quad \overline{\rho} \sim a^{-d\gamma} \, . \tag{3.17}$$

For $\gamma = -1/d$, which is the typical equation of state for a perfect gas of stretched (or unstable) strings [16], one thus recovers the particular solution already considered in [7,12] ("string-driven" pre-big-bang). More generally, however, the background (3.17) describes a phase of superinflationary expansion, $H > 0, \ddot{a}/a > 0$, and growing curvature scale, $\dot{H} > 0$, for all $\gamma < 0$.

This is the picture in the BD frame, which may be regarded as the natural one in a string theory context [3]. The passage to the E frame, defined as the frame in which the graviton and dilaton kinetic terms are diagonalized and the action takes the standard form,

$$S_E = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{|\tilde{g}|}[-\tilde{R}(\tilde{g}) + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}] + S_m \, , \tag{3.18}$$

is obtained through the conformal rescaling

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} e^{-2\phi/(d-1)} \quad , \quad \tilde{\phi} = \sqrt{\frac{2}{d-1}}\phi. \tag{3.19}$$

The E-transformed scale factor, $\tilde{a}$, and cosmic time coordinate, $\tilde{t}$, are thus related to the original BD ones by

$$\tilde{a} = a e^{-\phi/(d-1)} \quad , \quad d\tilde{t} = dt e^{-\phi/(d-1)}. \tag{3.20}$$

The pre-big-bang configuration (3.17) becomes, in the E frame,

$$\tilde{a}(\tilde{t}) \sim (-\tilde{t})^\beta \quad , \quad \tilde{\phi} \sim \sqrt{\frac{2}{d-1}} \frac{(d-1)(1-d\gamma)}{(\gamma-1)} \ln \tilde{a}$$

$$\tilde{\rho} \sim \tilde{a}^{-2/\beta} \quad , \quad \beta = \frac{2(1-\gamma)}{(d-1)(1+d\gamma^2) - 2(d\gamma - 1)} \tag{3.21}$$

where $\tilde{\rho}$ is conformally related to the original density ρ as

$$\tilde{\rho} = \rho \frac{\sqrt{|g|}}{\sqrt{|\tilde{g}|}} = \rho e^{\phi(d+1)/(d-1)} \tag{3.22}$$

(see for instance [17]). For all $d > 1$ and $\gamma < 0$, the transformed metric (3.21) satisfies

$$\frac{\ddot{\tilde{a}}}{\tilde{a}} < 0 \quad , \quad \tilde{H} < 0 \quad , \quad \dot{\tilde{H}} < 0 \, , \tag{3.23}$$

where $\tilde{H} = \dot{\tilde{a}}/\tilde{a}$, and the dot denotes here differentiation with respect to $\tilde{t}$. The BD superinflation thus becomes an accelerated contraction of the type (2.3).

This result is a consequence of the non-trivial evolution of the dilaton background that determines the transformation between the two frames, and it is of crucial importance. It implies that, if inflation is long enough in the BD frame to solve the kinematical problems of the standard model, then such problems are also solved in the E frame. Indeed, according to eq. (3.20), the two frames have the same conformal time

$$d\tilde{\eta} = \frac{d\tilde{t}}{\tilde{a}(\tilde{t})} = \frac{dt}{a(t)} = d\eta \tag{3.24}$$

and we have shown in Section 2 that the conditions to be satisfied for solving the kinematical problems, when expressed in conformal time, are the same for both superinflationary expansion and accelerated contraction.

Moreover, the spectrum of the metric perturbations amplified in the course of the background evolution is also the same in both frames. This can be easily shown by considering , for instance, the case of tensor perturbations, and assuming a generic model of background evolution characterized by the transition (at $\eta = \eta_1$) from the accelerated phase to the standard radiation-dominated one. In conformal time, such evolution can be parametrized as

$$\begin{aligned} a \sim (-\eta)^{-\delta} \quad &, \quad \phi \sim \epsilon \ln a \quad , \quad \eta << -\eta_1 \\ a \sim \eta \quad &, \quad \phi \sim const \quad , \quad \eta >> -\eta_1 \ . \end{aligned} \tag{3.25}$$

In order to verify the equality of the spectral behaviour, it is crucial to take into account the fact that not only the background solutions, but also the perturbation equations are different, when the frame is changed. In the BD frame, the tensor perturbation equation contains explicitly the contribution of the dilaton background, and for each component of h_μ^ν the equation can be written [7,17]

$$\psi'' + (k^2 - V)\psi = 0 \ , \tag{3.26}$$

where [7,17]

$$\psi = h a^{(d-1)/2} e^{-\phi/2}$$

$$V = \frac{(d-1)a''}{2a} - \frac{\phi''}{2} + \frac{(d-1)(d-3)a'^2}{4a^2} + \frac{\phi'^2}{4} - \frac{(d-1)a'\phi'}{2a} \ . \tag{3.27}$$

By matching the solutions of (3.26) corresponding to the two phases of background evolution, one can compute the Bogoliubov coefficients relating $|in\rangle$ and

$|out\rangle$ vacua, and describing the associated graviton production. For co-moving frequencies k that are small enough with respect to the height of the effective potential barrier ($k\eta_1 << 1$), the modulus of the Bogoliubov coefficient is [7,18]

$$|c_-(k)| \simeq (k\eta_1)^{-|\nu|-1/2} \tag{3.28}$$

where

$$\nu = \frac{\delta}{2}(d-1-\epsilon) + \frac{1}{2} \tag{3.29}$$

and the corresponding spectral distribution of gravitons is determined as $\rho(k) = k^4|c_-|^2$. In the case of four-dimensional exponential inflation ($\delta = 1, d = 3, \epsilon = 0$) one thus finds, in particular, the flat Harrison-Zeldovich spectrum.

In the more general case of the background (3.17), one finds that, in conformal time, the kinematics is parametrized according to eq. (3.25) by

$$\delta = -\frac{2\gamma}{1-2\gamma+d\gamma^2} \quad , \quad \epsilon = \frac{d\gamma-1}{\gamma} \; . \tag{3.30}$$

The coefficient $|\nu|$ determining the pre-big-bang graviton spectrum in the BD frame is thus

$$|\nu| = \frac{1}{2}\left|\frac{d\gamma^2-1}{1-2\gamma+d\gamma^2}\right| \; . \tag{3.31}$$

In the E frame, there is no explicit dilaton contribution to the perturbation equation for h, which is exactly the same equation as that satisfied by a minimally coupled scalar field [11] (the dilaton contribution, however, is implicitly contained in the rescaled metric background). Such an equation can still be written in the form (3.26), (3.27), but with $\phi = const.$ As a consequence, the spectral coefficient $|\nu|$ of eq. (3.28) is determined by the metric background only, and becomes

$$\nu = \frac{\tilde{\delta}}{2}(d-1) + \frac{1}{2} \; , \tag{3.32}$$

where $\tilde{\delta}$ is the exponent parametrizing, in conformal time, the evolution of the contracting E metric (3.21):

$$\tilde{\delta} = \frac{2(\gamma-1)}{(d-1)(1-2\gamma+d\gamma^2)} \; . \tag{3.33}$$

This value, when inserted into eq. (3.32), provides exactly the same expression for $|\nu|$ as in eq. (3.31), and thus the same graviton spectrum as in the BD frame.

We want to stress, finally, that the same results hold in the case of conformal vacuum backgrounds, namely for solutions of eqs. (3.5-3.7) with $\rho = p = 0$ [8,15]

(the general vacuum solution for the action (3.1) with non-zero $H_{\mu\nu\alpha}$ is given in [19]).

In the vacuum case the analogous of the isotropic, d-dimensional solution (3.17) is, in the BD frame,

$$\begin{aligned} a_{\mp}(t) &\sim |t|^{\mp 1/\sqrt{d}} \\ \phi_{\mp}(t) &\sim -(1 \pm \sqrt{d}) \ln |t| = \pm(\sqrt{d} \pm d) \ln a_{\mp} \end{aligned} \tag{3.34}$$

The two signs correspond to the two duality-related solutions [8], and the upper sign describes a "dilaton-driven", pre-big-bang, superinflationary expansion for t ranging from $-\infty$ to 0.

In the E frame the solution (3.34) becomes (in conformal time)

$$\tilde{a}(\tilde{\eta}) = |\tilde{\eta}|^{1/(d-1)} \quad , \quad \tilde{\phi}(\tilde{\eta}) = \mp\sqrt{2d(d-1)} \ln \tilde{a} \tag{3.35}$$

and it always describes an accelerated contraction of the type (2.3), independently of the choice of sign in eq. (3.34). It is interesting to note that the duality transformation, which is represented in the BD frame as an inversion of the scale factor and a related dilaton shift,

$$a_+ \to a_- = a_+^{-1} \quad , \quad \phi_+ \to \phi_- = \phi_+ - 2d \ln a_+ \tag{3.36}$$

becomes, in the E frame, a transformation between what we may call a strong-coupling and a weak-coupling regime, $\tilde{\phi} \to -\tilde{\phi}$, without changing the metric background described by $\tilde{a}$.

4. Conclusions

The main goal of this paper has been to show that, for what concerns the solution of the kinematical problems (horizon, flatness) of the standard model, and the amplification of the vacuum fluctuations, an accelerated contraction of the metric is equally good as an accelerated expansion.

This observation was motivated by the fact (also discussed in this paper) that accelerated contraction is the behaviour of the metric in a general pre-big-bang cosmological string scenario, when seen in the Einstein frame. Indeed, as already stressed in [7], there are only two ways of implementing a phase of cosmic acceleration and simultaneous growth of the curvature scale: accelerated contraction and superinflationary (or pole-like) expansion. The latter corresponds to the pre-big-bang picture in the conformally related Brans-Dicke frame.

Obviously, a contracting phase cannot dilute the relic abundance of some unwanted remnants, such as the monopoles of the GUT phase transition. However, the same is true for the pre-big-bang scenario in the BD frame, as well as for all models in which the phase of inflationary expansion occurs at some higher fundamental (near Planckian) scale, which is indeed what is expected in a string cosmology context. In this respect, we recall [7] that a pre-big-bang phase should be regarded not necessarily as an alternative, but possibly as a complement to the more conventional inflationary models, which cannot be extended (at least semiclassically) beyond the Planck era.

Moreover, it is clear that deflationary contraction is adiabatic for what concerns radiation, just like the usual inflationary expansion. Therefore, as recently stressed also in [20], a kinematical modification of the standard model can explain the large present value of the cosmic black-body entropy, only if the accelerated evolution is matched to the standard one through a phase dominated by some non-adiabatic process (the so-called "reheating" era).

In the BD picture of the pre-big-bang scenario (see eq. (3.17)), the radiation is supercooled and diluted with respect to the sources that drive inflation. The conservation equation (3.8) leads in fact to an effective source temperature $T_s \sim a^{-d\gamma}$, which grows together with the scale factor for $\gamma < 0$, and satisfies

$$\frac{T_s}{T_r} = \frac{\rho_\gamma}{\rho_r} \sim a^{1-d\gamma} \tag{4.1}$$

(r corresponds here to the radiation-like equation of state, $\gamma = 1/d$). The reheating process is thus expected to represent, in this frame, a sort of non-adiabatic conversion of the hot sources into radiation, such as a possible isothermal decay of the highly excited states of a gas of stretched strings [7].

In the E frame (see eq. (3.21)) the fluid sources satisfy a modified conservation equation,

$$\dot{\tilde{\rho}} + d\tilde{H}(\tilde{\rho} + \tilde{p}) - \frac{\dot{\tilde{\phi}}}{\sqrt{2(d-1)}}(\tilde{\rho} - d\tilde{p}) = 0 \ . \tag{4.2}$$

Radiation still evolves adiabatically, now with a blue-shifted temperature because of the contraction, $\tilde{T}_r \sim \tilde{a}^{-1}$. The effective temperature of the pre-big-bang sources is also blue-shifted, however, since, in the perfect fluid approximation, eq. (4.2) leads to

$$\tilde{T}_s \sim \tilde{a}^{(d^2\gamma^2+1-d\gamma-d\gamma^2)/(\gamma-1)} \tag{4.3}$$

and thus

$$\frac{\tilde{T}_s}{\tilde{T}_r} = \frac{\tilde{\rho}_s}{\tilde{\rho}_r} \sim \tilde{a}^{\gamma(d-1)(d\gamma-1)/(\gamma-1)} \ . \tag{4.4}$$

For $\tilde{a} \to 0$ the temperature of the sources that drive the acceleration ($\gamma < 0, d > 1$) is always growing, even with respect to the radiation temperature. The physical picture of reheating as a non-adiabatic decay of the hot sources is still valid, therefore, also in the Einstein frame.

We would like to stress, finally, that the absence of problems related to some "preferred frame" description of a string cosmology inflation is to be ascribed, to a large extent, to the crucial role played by the dilaton field, which transforms conformally a superinflationary expansion into a deflationary contraction. This is to be traced back to the duality properties of the string effective action [8,12,19,21], and thus gives support to the consistency of an approach to string cosmology based on the effective action (3.1).

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The String Dilaton and a Least Coupling Principle

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Abstract

It is pointed out that string-loop modifications of the low-energy matter couplings of the dilaton may provide a mechanism for fixing the vacuum expectation value of a massless dilaton in a way which is naturally compatible with existing experimental data. Under a certain assumption of universality of the dilaton coupling functions , the cosmological evolution of the graviton-dilaton-matter system is shown to drive the dilaton towards values where it decouples from matter ("Least Coupling Principle"). Quantitative estimates are given of the residual strength, at the present cosmological epoch, of the coupling to matter of the dilaton. The existence of a weakly coupled massless dilaton entails a large spectrum of small, but non-zero, observable deviations from general relativity. In particular, our results provide a new motivation for trying to improve by several orders of magnitude the various experimental tests of Einstein's Equivalence Principle (universality of free fall, constancy of the constants,...).

I. INTRODUCTION

At present we know only one theory which treats gravity in a way consistent with quantum mechanics: string theory. In the low energy limit (low in comparison with the Planck mass) string theory gives back classical general relativity, with, however, an important difference. All versions of string theory predict the existence of a (four-dimensional) scalar partner of the tensor Einstein graviton: the dilaton. It may happen that this scalar field acquires a mass due to some yet unknown dynamical mechanism. This is the generally adopted view, and if so there will be no observable macroscopic difference between string gravity and Einstein gravity. In this paper we will discuss another possibility: that the dilaton remains massless. This immediately leads to the dramatic conclusion that all coupling constants and masses of elementary particles, being dependent on the dilaton scalar field, should be, generally speaking, space and time dependent, and influenced by local circumstances. This conclusion is of course not new and it was precisely the reason for discarding

the possibility that we are going to discuss. Indeed, it has been stated that the existence of a massless dilaton contributing to macroscopic couplings would, at once, entail the following observable consequences: (i) Jordan-Fierz-Brans-Dicke-type [1] deviations from Einstein's theory in relativistic $[O(Gm/c^2r)]$ gravitational effects [2]; (ii) cosmological variation of the fine structure constant, and of the other gauge coupling constants [3], and (iii) violation of the (weak) equivalence principle [4]. As the strength of the coupling of the dilaton to matter is expected to be comparable to that of the (spin 2) graviton, and even larger than it in the case of hadrons [4], the above observable consequences seem to be in violent conflict with experiment. Indeed, present experimental data give upper limits of order: (i) 10^{-3} for a possible fractional admixture of a scalar component to the relativistic gravitational interaction [5], (ii) $10^{-15}\mathrm{yr}^{-1}$ for the fractional variation with time of the fine-structure constant[1] [6], (iii) 10^{-11} — 10^{-12} on the universality of free fall (weak equivalence principle) [7–10] [2]. [See [11,12] for reviews of the comparison between gravitational theories and experiments].

In this paper, we point out that non-perturbative string loop effects (associated with worldsheets of arbitrary genus in intermediate string states) can naturally reconcile the existence of a massless dilaton with existing experimental data if they exhibit the same kind of universality as the tree level dilaton couplings. By studying the cosmological evolution of general graviton-dilaton-matter systems we show that the dilaton is cosmologically attracted toward values where it decouples from matter, a situation which we call the "Least Coupling Principle". Roughly speaking, the origin of the attraction is the following. Masses of different particles depend on the dilaton, while the source for the dilaton is the gradient of these masses. It is therefore not surprising to have a fixed point where the gradient of the masses is zero. [With some important differences discussed below, this mechanism is similar to the generic attractor mechanism of metrically-coupled tensor-scalar theories discussed in Refs. [13]]. This cosmological attraction is so efficient that the presently existing experimental limits do not place any significant constraints on the physical existence of a massless dilaton. Most importantly, we give quantitative estimates for the level of residual deviation from Einstein's theory expected at the present cosmological epoch, notably for the violation of the equivalence principle.

II. THE GRAVITON-DILATON-MATTER SYSTEM

At the tree level in the string loop expansion (spherical topology for intermediate worldsheets) the effective action describing the massless modes (here considered directly in four dimensions) has the general form [14–16]

$$S_{\rm tree} = \int d^4x\sqrt{\hat{g}}e^{-2\Phi}\{(\alpha')^{-1}[\hat{R} + 4\,\hat{\Box}\Phi - 4(\widehat{\nabla}\Phi)^2]$$

[1]Note that, within the QCD framework, it does not make sense to speak of the variation of any strong-interaction coupling constant (the hadron mass-scale adjusting itself such that $\alpha_{\rm strong} \simeq 1$).

[2]The most recent analysis of Lunar Laser Ranging data [10] finds that the fractional difference in gravitational acceleration toward the Sun between the (silica-dominated) Moon and the (iron-dominated) Earth is $(-2.7 \pm 6.2) \times 10^{-13}$.

$$-\frac{k}{4}\widehat{F}^a_{\mu\nu}\widehat{F}^{a\mu\nu} - \overline{\widehat{\psi}}\widehat{D}\widehat{\psi} + \cdots$$
$$+\sum_{n\geq 1} O[(\alpha'\partial^2)^n]\} \ . \tag{2.1}$$

Here, $\hat{g}_{\mu\nu}$ (often denoted $G_{\mu\nu}$) denotes the metric appearing in the σ-model formulation of string theory and is used for defining all the covariant constructs entering Eq. (1) $[\widehat{\nabla}, (\widehat{F})^2, \widehat{D}, \cdots]$; Φ denotes the dilaton; a summation over the various possible gauge fields $[F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu]$ and fermions $[\widehat{D} = \widehat{\gamma}^\mu(\widehat{\nabla}_\mu + A^a_\mu t^a)]$ is understood; the ellipsis stand in particular for the ill-understood remaining scalar sector of the theory [Higgs fields and their Yukawa couplings, and possibly other gauge-neutral scalar (moduli), or pseudo-scalar (axion,...), fields]; and the last term symbolically denotes the infinite series of higher-derivative terms representing the low-energy effects of all the massive string modes on which one has to integrate to get the effective action for the massless modes.

The remarkable feature that, when formulating the action in terms of the "string frame" metric $\hat{g}_{\mu\nu}$, the dilaton couples, at the string tree level, in a universal, multiplicative manner to all the other fields derives from the fact that $g_s \equiv \exp(\Phi)$ plays the role of the string coupling constant. In the σ-model formulation, this is easily seen to follow from applying the Gauss-Bonnet theorem $[(4\pi)^{-1}\int d^2\xi\sqrt{h}R^{(2)}(h) = \chi = 2(1-n)$; n= number of handles] to the Fradkin-Tseytlin σ-model dilaton term, $S_{\rm dil} = (4\pi)^{-1}\int d^2\xi\sqrt{h}\Phi(X)R^{(2)}$. In a constant (or slowly varying) dilaton background, the genus-n string-loop contribution to any string transition amplitude contains the factor $\exp(-S_{\rm dil}) = \exp(2(n-1)\Phi) = g_s^{2(n-1)}$. Therefore, when taking into account the full string loop expansion, the effective action for the massless modes will take the general form

$$S = \int d^4x\,\sqrt{\hat{g}}\Big\{\frac{B_g(\Phi)}{\alpha'}\,\widehat{R} + \frac{B_\Phi(\Phi)}{\alpha'}\,[4\hat{\Box}\Phi - 4(\widehat{\nabla}\Phi)^2]$$
$$-B_F(\Phi)\frac{k}{4}\widehat{F}^2 - B_\psi(\Phi)\overline{\widehat{\psi}}\widehat{D}\widehat{\psi} + \cdots\Big\} \ . \tag{2.2}$$

At this stage of development of string theory, one does not know how to control the structure of the various dilaton coupling functions $B_i(\Phi)$ $(i = g, \Phi, F, \psi, ...)$ beyond the fact that in the limit $\Phi \to -\infty$ $(g_s \to 0)$ they should admit an expansion in powers of $g_s^2 = \exp(+2\Phi)$ of the form,

$$B_i(\Phi) = e^{-2\Phi} + c_0^{(i)} + c_1^{(i)}e^{2\Phi} + c_2^{(i)}e^{4\Phi} + \cdots \tag{2.3}$$

[Note that we have in mind the low-energy regime , with broken supersymmetry, for which there are no *a priori* obstacles to having couplings of the type (2.3) with $c_n^{(i)} \neq 0$.]

Concerning the low-energy effects of all the massive string modes, we shall assume for simplicity that, like at tree level, Eq. (2.1), they are equivalent to introducing a cut-off at a Φ-independent string mass scale $\widehat{\Lambda}_s \sim (\alpha')^{-1/2}$, when measuring distances by means of the string-frame metric $\hat{g}_{\mu\nu}$.

It is convenient to transform the action (2.2) by introducing several Φ-dependent rescalings. One can put both the gravity and the fermion sectors into a standard form by: (i) introducing the "Einstein metric",

$$g_{\mu\nu} \equiv C\ B_g(\Phi)\hat{g}_{\mu\nu} \tag{2.4}$$

(with some numerical constant C[3]), (ii) replacing the original dilaton field Φ by the variable[4]

$$\varphi \equiv \int d\Phi \left[\frac{3}{4}\left(\frac{B_g'}{B_g}\right)^2 + 2\frac{B_\Phi'}{B_g} + 2\frac{B_\Phi}{B_g}\right]^{1/2} \tag{2.5}$$

(where a prime denote $d/d\Phi$), and (iii) rescaling the Dirac fields

$$\psi \equiv C^{-3/4} B_g^{-3/4} B_\psi^{1/2} \widehat{\psi}\ . \tag{2.6}$$

The transformed action can be decomposed into a gravity sector $(g_{\mu\nu}, \varphi)$ and a matter one $(\psi, A, \cdots)$

$$S[g, \varphi, \psi, A, \cdots] = S_{g,\varphi} + S_m\ , \tag{2.7a}$$

$$S_{g,\varphi} = \int d^4x\, \sqrt{g} \left\{\frac{1}{4q}R - \frac{1}{2q}(\nabla\varphi)^2\right\}\ , \tag{2.7b}$$

$$S_m = \int d^4x\, \sqrt{g} \left\{-\overline{\psi}\, D\, \psi - \frac{k}{4}\, B_F(\varphi) F^2 + \cdots\right\}\ . \tag{2.7c}$$

Here, $q \equiv 4\pi\overline{G} \equiv \frac{1}{4}C\alpha'$ ($\overline{G}$ denoting a bare gravitational coupling constant), $B_F(\varphi) \equiv B_F[\Phi(\varphi)]$ and the ellipsis stand for the (more complicated) Higgs sector. One should note that the string cut-off mass scale acquires a dependence upon the dilaton in Einstein units:

$$\Lambda_s(\varphi) \equiv C^{-1/2} B_g^{-1/2}(\varphi) \widehat{\Lambda}_s\ . \tag{2.8}$$

Essential to the following will be the dilaton dependence of the matter Lagrangian. One does not know at present how to relate string models to the observed particle spectrum. The basic clue that we shall follow is the dilaton dependence of the gauge coupling constants: $g^{-2} = k\, B_F(\varphi)$ from (2.7c). To connect the (bare) effective action (2.7) (integrated over the massive string modes) to the low-energy world, one still needs to take into account the quantum effects of the light modes between the string scale $\Lambda_s(\varphi)$ and some observational scale. In the case of an asymptotically free theory the ratio of the IR confinment mass scale $\Lambda_{\rm conf}$ to the cut-off scale, is, at the one-loop level, exponentially related to the inverse of the gauge coupling constant appearing in the bare action:

$$\Lambda_{\rm conf} \sim \Lambda_s \exp(-8\pi^2 b^{-1} g^{-2}) = C^{-1/2} B_g^{-1/2}(\varphi) \exp[-8\pi^2 b^{-1} k\, B_F(\varphi)] \widehat{\Lambda}_s\ , \tag{2.9}$$

where the one-loop coefficient b depends upon the considered gauge field as well as the matter content. The mass of hadrons is, for the most part, generated by QCD-effects and is simply

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[3]We shall choose C such that the string units and the Einstein units coincide at the present cosmological epoch: $CB_g(\Phi_0) = 1$.

[4]When $B_g = B_\Phi$ the quantity under the square root in Eq.(2.5) is positive definite.

proportional to Λ_{QCD} (with some pure number as proportionality constant). The dilaton dependence of the QCD part of the mass of hadrons is therefore given by (2.9) with $b = b_3$ and $B_F = B_3$ being the appropriate QCD quantities. However, the lepton masses, and the small quark contributions to the mass of hadrons, are not related to Λ_{QCD} (at least in any known way). Their dilaton dependence is defined by specific mechanisms of spontaneous symmetry breakdown (and compactification) which depend on particular string models and are not well established at present. Let us note that in technicolor-type models, as well as in no-scale supergravity ones, all the particle mass scales are related to the fundamental cut-off scale by formulas of the type (2.9).

As a minimal ansatz, we can assume that the mass (in Einstein units) of any type of particle, labelled A, depends in a non-trivial way on the VEV of the dilaton through some of the functions B_i appearing in (2.2):

$$m_A(\varphi) = m_A[B_g(\varphi),\ B_F(\varphi), \cdots] \ . \tag{2.10}$$

The essential new feature allowed by nonperturbative string-loop effects (i.e. arbitrary functions $B_i(\Phi)$, Eq. (2.3)) is the possibility for the function $m_A(\varphi)$ to admit a minimum for some finite value of φ. Assuming this, we shall see below that the cosmological evolution naturally attracts φ to such a minimum. However, if the various coupling functions $B_i(\varphi)$ differ from each other the minima of $m_A(\varphi)$ depend , in general, on the type of particle considered. It will be seen below that this weakens the attraction effect of the cosmological expansion,and, more importantly, leaves room for violations of the equivalence principle at a probably unacceptable level (see the footnote following Eq.(6.13)). This suggests to concentrate on the case where string-loop effects preserve the universal multiplicative coupling present at tree-level, Eq.(2.1), i.e. the case where all the dilaton coupling functions coincide: $B_i(\varphi) = B(\varphi)$ for $i = F, g, \cdots$. In this "universal $B(\varphi)$" case, the extrema of the function $m_A(\varphi) = m_A[B(\varphi)]$ will (generically) coincide with the extrema of the function $B(\varphi)$. As discussed below, this assumption leads very naturally (without fine-tuning, or the need to inject small parameters) to a situation where the present deviations from general relativity are so small as to have escaped detection. When we shall need in the following to estimate quantitatively the dependence of particle masses on φ, we shall assume that the mass of any particle A is of the form suggested by Eq.(2.9):

$$m_A(\varphi) = \mu_A B^{-1/2}(\varphi) \exp[-8\pi^2 \nu_A B(\varphi)] \widehat{\Lambda}_s \ , \tag{2.11}$$

with μ_A and ν_A pure numbers of order unity. We believe that our main qualitative conclusions do not depend strongly on the specific form of the assumption (2.11).

For the quantitative estimates below we need to choose some specific value of the string unification scale $\widehat{\Lambda}_s \propto \alpha'^{-1/2}$. The theoretical value $\widehat{\Lambda}_s = e^{(1-\gamma)/2} 3^{-3/4} g_s M_{\rm Planck}/4\pi \simeq g_s \times 5.27 \times 10^{17}$GeV has been suggested [17]. Here g_s denotes the common (modulo possible Kac-Moody level factors of order unity) value of the gauge coupling constants at the string scale. To fix ideas, we shall take $\widehat{\Lambda}_s = 3 \times 10^{17}$GeV.

III. CLASSICAL COSMOLOGY WITH A DILATON

The gravitational field equations derived from Eqs. (2.7) read

$$R_{\mu\nu} = 2\partial_\mu\varphi\,\partial_\nu\varphi + 2q\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) \ , \tag{3.1a}$$

$$\Box\varphi = -q\sigma \ , \tag{3.1b}$$

where the source terms, defined by $T^{\mu\nu} \equiv 2g^{-1/2}\delta S_m/\delta g_{\mu\nu}$, $\sigma \equiv g^{-1/2}\delta S_m/\delta\varphi$, are related by the energy balance equation: $\nabla_\nu T^{\mu\nu} = \sigma\nabla^\mu\varphi$.

In the case of a Friedmann cosmological model, $ds^2 = -dt^2 + a^2(t)d\ell^2$ with $d\ell^2 = (1-Kr^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$, $K = 0$, $+1$ or -1, the field equations give ($T^\mu_\nu = \mathrm{diag}(-\rho, P, P, P)$; $H \equiv \dot{a}/a$, the overdot denoting d/dt)

$$-3\frac{\ddot{a}}{a} = q(\rho + 3P) + 2\dot{\varphi}^2 \ , \tag{3.2a}$$

$$3H^2 + 3\frac{K}{a^2} = 2q\rho + \dot{\varphi}^2 \ , \tag{3.2b}$$

$$\ddot{\varphi} + 3H\,\dot{\varphi} = q\,\sigma \ . \tag{3.2c}$$

In the following, we concentrate on the spatially flat case ($K = 0$). Following Ref. [13], we can combine Eqs. (3.2) to write a simple equation for the cosmological evolution of the dilaton with respect to the logarithm of the cosmological scale factor: $p \equiv \ln(a) + \text{const.}$ (not to be confused with the pressure P). Denoting d/dp by a prime, one gets ($K = 0$)

$$\frac{2}{3 - \varphi'^2}\varphi'' + (1-\lambda)\varphi' = \frac{\sigma}{\rho} \ , \tag{3.3}$$

where $\lambda \equiv P/\rho$.

Except during phase transitions, the material content of the universe can be classically described as a superposition of several (weakly interacting) gases labelled by A, i.e. by an action of the form

$$S_m[g, \varphi, x_A] = -\sum_A \int m_A[\varphi(x_A)][-g_{\mu\nu}(x^\lambda_A)dx^\mu_A dx^\nu_A]^{1/2} \tag{3.4}$$

(the massless particles being obtained by taking the limit $m_A \to 0$ with $m_A u^\mu_A \equiv m_A dx^\mu_A/ds_A$ fixed). In Eq. (3.4) the summation over A includes a sum over the statistical distribution of the A-type particles. The gravitational source terms corresponding to Eq. (3.4) read

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{g(x)}}\sum_A \int ds_A m_A[\varphi(x_A)]u^\mu_A u^\nu_A \delta^{(4)}(x - x_A) \ , \tag{3.5a}$$

$$\begin{aligned}\sigma(x) &= -\frac{1}{\sqrt{g(x)}}\sum_A \int ds_A \alpha_A[\varphi(x_A)]m_A[\varphi(x_A)]\delta^{(4)}(x - x_A) \\ &= \sum_A \alpha_A[(\varphi(x)]\ T_A(x) \ ,\end{aligned} \tag{3.5b}$$

where

$$\alpha_A(\varphi) \equiv \frac{\partial \ln\, m_A(\varphi)}{\partial \varphi} \tag{3.6}$$

measures the strength of the coupling of the dilaton to the A-type particles. In the second Eq. (3.5b) $T_A = -\rho_A + 3P_A$ denotes the trace of the A-type contribution to the total $T^{\mu\nu} = \Sigma_A T_A^{\mu\nu}$. It is easy to see that when the different A-gases are non interacting their corresponding sources satisfy the separate energy balance equations: $\nabla_\nu T_A^{\mu\nu} = \sigma_A \nabla^\mu \varphi = \alpha_A T_A \nabla^\mu \varphi$.

In the string context, it is natural to assume that the string scale Λ_s subsumes both what is usually meant by "Planck scale" and "GUT scale", leaving essentially no room for a quasi-classical inflationary era. We leave to future work a discussion of primordial stringy cosmology, and content ourselves by describing the evolution of dilatonic cosmologies through a radiation-dominated era, followed by a matter-dominated one.

IV. EVOLUTION OF THE DILATON DURING THE RADIATION-DOMINATED ERA.

During a radiation-dominated era (universe dominated by ultra-relativistic gases) the gravitational source terms are approximately given by

$$\rho \simeq 3P \simeq g_*(T) \frac{\pi^2}{30} T^4 \ , \tag{4.1a}$$

$$\sigma \simeq 0 \ , \tag{4.1b}$$

where $g_*(T) = \sum_{\rm Bose} g_A^B (T_A/T)^4 + (7/8) \sum_{\rm Fermi} g_A^F (T_A/T)^4$ is the effective number of relativistic degrees of freedom in the cosmic soup at temperature T. [The sum defining $g_*(T)$ is taken only over particles with mass $m_A \ll T$; because of possible previous decouplings the corresponding relativistic gases may not all have the temperature T, e.g. $T_\nu = (4/11)^{1/3} T_\gamma$ below 1 MeV]. Eq. (4.1b) suggests that the dilaton does not evolve during the radiation era. More precisely, Eq. (3.3) with $\lambda \simeq 1/3$ shows that $\varphi(p)$ behaves as a particle, with velocity-dependent mass, submitted to a constant friction. In a few p-time units, $\varphi(p)$ will exponentially come to rest. [see Ref. [13] for the exact solution of the damped evolution of $\varphi(p)$ when σ/ρ is negligible]. However, something interesting happens each time the universe cools down to a temperature $T \sim m_A$ defining the threshold for the participation of the species A to the relativistic soup. When $T \sim m_A$, the term on the right-hand side of the p-time evolution of φ is well approximated by

$$\frac{\sigma_A}{\rho_{\rm tot}} = -\frac{15}{\pi^4} \frac{g_A}{g_*(T)} \tau_\pm(z_A) \alpha_A(\varphi) \ , \tag{4.2}$$

where $z_A \equiv m_A/T$ and

$$\tau_\pm(z) \equiv z^2 \int_z^\infty dx \ \frac{(x^2 - z^2)^{1/2}}{e^x \pm 1} \ , \tag{4.3}$$

where the upper (lower) sign corresponds to A being a fermion (boson). In the approximation (justified by the results to be discussed) where the dilaton contributions to the Einstein

equations (3.2a), (3.2b) are negligible one has $T \propto a^{-1}$, and therefore $p = \ln z_A$ (with an adapted choice of origin for p). Then, as a function of p, the (everywhere positive) function $\tau_\pm$ is proportional to $\exp(+2p)$ when $p \to -\infty$, rises up to a maximum $\simeq 1.16$ when $z_A \simeq 0.87$, and falls quickly to zero as $\exp[\frac{5}{2}p - \exp(p)]$ when $p \to +\infty$. [This maximum occurs because for $T \gg m_A$ the (ultra-relativistic) particles do not contribute to σ, while for $T \ll m_A$ there are exponentially few particles]. Remembering the definition (3.6) of $\alpha_A(\varphi)$, it is easy to see that if the function $m_A(\varphi)$ has a minimum, say φ_m^A, and if the initial value of φ, say $\varphi_-^A \equiv \varphi(p = -\infty) = \varphi(T \gg m_A)$, is sufficiently near φ_m^A, Eq. (3.3) will describe a damped, transient nonlinear attraction of $\varphi(p)$ around φ_m^A. [Note that, from the above discussion, the initial velocity is zero to an exponential accuracy $\sim \exp(p_{A'} - p_A)$]. The existence of such an attraction mechanism by mass thresholds during the radiation era was noticed in Ref. [13] in a related context (generalized Jordan-Fierz-Brans-Dicke theories characterized by a universal, A-independent coupling function $\alpha_A(\varphi) = \alpha(\varphi) = \partial a(\varphi)/\partial\varphi$). In the context of Ref. [13], it seemed natural to assume that the curvature of the function $\ln m(\varphi) = a(\varphi) + \ln m_0$ near its minimum was of order unity. This rendered the presently discussed attraction mechanism very ineffective. An important new feature of the present, dilatonic, context is that the curvature of $\ln m_A(\varphi)$ near its minimum is expected to be large compared to one. This follows from the expected exponential dependence on $B_F(\varphi)$ of the mass scales of the low-energy particle spectrum, Eq. (2.9). To fix ideas and be able to make some quantitative estimates, we shall take the form (2.11) with $\mu_A = 1$. This yields

$$\alpha_A(\varphi) = -\left[\ln\frac{\widehat{\Lambda}_s}{m_A} + \frac{1}{2}\right]\frac{\partial \ln B(\varphi)}{\partial\varphi} = +\ln\frac{\widehat{\Lambda}'_s}{m_A}\,\frac{\partial \ln B^{-1}(\varphi)}{\partial\varphi}\ , \tag{4.4}$$

where $\widehat{\Lambda}'_s = e^{1/2}\widehat{\Lambda}_s \simeq 5 \times 10^{17}\mathrm{GeV}$.

We see that a minimum φ_m of $m_A(\varphi)$ corresponds to a maximum of $B(\varphi)$ (or a minimum of $B^{-1}(\varphi)$). Let us denote by κ the curvature of the function $\ln B^{-1}(\varphi)$ near its minimum φ_m. In the parabolic approximation

$$\ln B^{-1}(\varphi) \simeq \ln B^{-1}(\varphi_m) + \frac{1}{2}\kappa(\varphi - \varphi_m)^2\ , \tag{4.5}$$

one gets

$$\alpha_A(\varphi) = \beta_A(\varphi - \varphi_m)\ , \tag{4.6a}$$

$$\beta_A = \kappa\,\ln(\widehat{\Lambda}'_s/m_A) = \kappa[40.75 - \ln(m_A/1\mathrm{GeV})]\ . \tag{4.6b}$$

Inserting Eq. (4.6a) into Eq. (4.2) and then into Eq. (3.3) (written in the approximation $\lambda \simeq 1/3$) yields

$$\varphi''(p) + \varphi'(p) = s_A(p)[\varphi(p) - \varphi_m]\ , \tag{4.7}$$

with

$$s_A(p) = -\frac{45}{2\pi^4}\,\beta_A\frac{g_A}{g_{*(T)}}\ \tau_\pm(e^p)\ . \tag{4.8}$$

Within a good approximation one can replace the temperature-dependent quantity $g_A/g_*(T)$ by its initial value, say $f_A^{\rm in} \equiv g_A/g_*^{\rm in}$, in which $g_*^{\rm in} \equiv g_*(T \gg m_A)$ contains the contribution $7g_A/8$ (or g_A) if A is a fermion (or boson). Eq. (4.7) describes a damped motion submitted to a transient harmonic force tending to attract φ toward φ_m. The final outcome of this motion is to leave (when $p = +\infty$) φ nearer to φ_m than it was when it started at rest at $p = -\infty$. We define the attracting factor of the A-th mass threshold as $m_\pm(b_A) \equiv (\varphi(+\infty) - \varphi_m^A)/(\varphi(-\infty) - \varphi_m^A)$, where the suffix $\pm$ in the left-hand side corresponds to the fermion/boson case and where $b_A \equiv \beta_A f_A^{\rm in} \equiv \beta_A g_A/g_*^{\rm in}$. There are two quite different regimes in this mass-threshold attraction mechanism: when $b_A \ll 1$ ($b_A < 0.5$ sufficing), $\varphi(p)$ moves monotonically toward φ_m by a small amount given by integrating over p the force term on the right-hand side of (4.7) evaluated at the original position of φ ("kick" approximation). The result is (see Ref. [13])

$$m_\pm(b_A) = 1 - \frac{1}{2} b_A \begin{pmatrix} 7/8 \\ 1 \end{pmatrix} + O(b_A^2) \ , \tag{4.9}$$

where the upper (lower) coefficient corresponds to the fermion (boson) case, respectively. In this first case the attracting power of the A-threshold is rather weak (hence the conclusion of Ref. [13] that the total radiation era attraction is rather ineffective in the case of usual tensor-scalar theories with $\beta_A = O(1)$ and $\Sigma_A f_A \sim \ln(100/10) \simeq 2.3$). By contrast, in the present, dilatonic context one expects $\kappa \sim 1$, $\beta_A \sim 40$ and therefore $b_A \gg 1$ for many mass thresholds (the most efficient mass thresholds being the latest in the radiation era which tend to have the largest $f_A^{\rm in}$'s: notably the e^+e^- threshold with $f_e^{\rm in} = 4/10.75 \simeq 0.372$). When $b_A \gg 1$ ($b_A > 2$ sufficing in practice) one can analytically solve Eq. (4.7) by a WKB-type approach. [With some subtleties compared to the usual WKB approximation as the matching between the damped and oscillating regions must be done via Bessel functions instead of the usual Airy ones]. Qualitatively the motion of $\varphi(p)$ begins by a slow roll toward φ_m, continues by WKB oscillations around φ_m, and terminates as a damped inertial motion. The final analytical results for the attraction factor reads ($b_A \gg 1$)

$$m_\pm(b_A) = (C_\pm b_A)^{-1/4} \cos \theta_\pm^A \ , \tag{4.10a}$$

with $C_+ = 15/8$, $C_- = 15/4$ and

$$\theta_\pm^A = \int_{-\infty}^{+\infty} [-s_A(p)]^{1/2} dp - \frac{\pi}{4} = b_A^{1/2} I_\pm - \frac{\pi}{4} \ , \tag{4.10b}$$

with $I_+ \simeq 1.2743$, $I_- \simeq 1.4029$. Note that when $b_A \to \infty$, $|m_\pm(b_A)|$ tends to zero as $O(b_A^{-1/4})$. Fig. 1 represents the two functions $m_\pm(b)$, obtained by numerically integrating Eq. (4.7).

One must take into consideration the fact that mass thresholds can occur only for particles whose masses are smaller than the critical temperature of the phase transition through which they acquired a mass (e.g. the pions are the only hadrons to take into account). The Higgs threshold is to be considered as part of the electroweak phase transition, and the strange quark threshold overlaps with the quark-hadron phase transition. This leaves nine, clearly present, mass thresholds associated (in decreasing temperature scale) with the top quark ($f_t^{\rm in} = 12/106.75$, $\beta_t \simeq 35.74\kappa$), the Z^0 ($f_Z^{\rm in} = 3/95.25$, $\beta_Z \simeq 36.24\kappa$), the $W^\pm$ ($f_W^{\rm in} = 6/92.25$, $\beta_W \simeq 36.37\kappa$), the bottom quark ($f_b^{\rm in} = 12/86.25$, $\beta_b \simeq 39.14\kappa$), the tau

($f_\tau^{\rm in} = 4/75.75$, $\beta_\tau \simeq 40.17\kappa$), the charmed quark ($f_c^{\rm in} = 12/72.25$, $\beta_c \simeq 40.35\kappa$), the pions ($f_\pi^{\rm in} = 3/17.25$, $\beta_\pi \simeq 42.74\kappa$), the muon ($f_\mu^{\rm in} = 4/14.25$, $\beta_\mu \simeq 43.00\kappa$), and the electron ($f_e^{\rm in} = 4/10.75$, $\beta_e \simeq 48.33\kappa$). The quoted values of $f_A^{\rm in}$ and β_A show that $b_A \equiv \beta_A f_A^{\rm in}$ is typically a few times κ (with extreme values 1.14κ and 17.98κ for the Z and e respectively). A look at Fig. 1 shows immediately that if κ is of order unity, each mass threshold will a be rather efficient attractor. The compound effect of all those attractors is discussed below.

Besides mass thresholds, phase transitions provide another possible attractor mechanism for the dilaton during the radiation-dominated era. During a phase transition the vacuum energy density V changes from some positive value, say $V^{\rm in} = g_{\rm vac}(\pi^2/30)T_c^4$, when $T > T_c$ to a comparatively negligible value when $T < T_c$. For instance, in the case of the QCD (quark-hadron) phase transition one has $T_c \simeq 200$ MeV and $g_{\rm vac} = 34/3$ (in a simple model [18] describing the unconfined phase as a relativistic gas of gluons and u and d quarks — besides γ, e, ν and μ — and the confined phase as a relativistic gas of pions). Besides its dependence on the temperature the vacuum energy density is also a function of the dilaton. Therefore the vacuum term in the matter action, $S_{\rm vac} = -\sqrt{g}V(\varphi, T)$, will generate a corresponding source term $\sigma_{\rm vac} = -\partial V/\partial\varphi$ in the right-hand sides of the dilaton evolution equations (3.1b), (3.2c) or (3.3). In the simple model of the QCD phase transition just described, one can estimate the source term $\sigma_{\rm vac}$ by assuming that the dilaton dependence of V is essentially contained in the φ-dependence of the critical temperature T_c. In turn, the latter dependence is obtained from $T_c \sim \Lambda_{QCD}$ with $\Lambda_{QCD}(\varphi)$ given by Eq. (2.9) with the appropriate one-loop coefficient. This shows that φ will be attracted toward a minimum of $\Lambda_{QCD}(\varphi)$. More precisely, if we assume, to fix ideas, that $B_g(\varphi) = B_F(\varphi)$ in Eq. (2.9) and that φ is near the maximum φ_m of $B(\varphi)$, one gets, in the parabolic approximation (4.5) (setting as above $\lambda = P_{\rm tot}/\rho_{\rm tot}$ and $p = \ln(T_c/T)$)

$$\varphi''(p) + \frac{3}{2}[1 - \lambda(p)]\varphi'(p) = s_{\rm vac}(p)[\varphi(p) - \varphi_m] \ , \tag{4.11}$$

where $s_{\rm vac}(p) \simeq -6\beta_{\rm vac} f_{\rm vac} \exp(4p)$ when $p \to -\infty$, with $\beta_{\rm vac} = \kappa \ln(\widehat{\Lambda}'_s/T_c)$ and $f_{\rm vac} = g_{\rm vac}/g_*(T > T_c)$. After the phase transition, when $p \to +\infty$, one expects $s_{\rm vac}(p)$ to fall quickly to zero as $\exp(-aT_c/T) = \exp(-a\exp(p))$ with a of order unity. In the limit where $b_{\rm vac} \equiv \beta_{\rm vac} f_{\rm vac}$ is large enough to make φ oscillate around φ_m, one can solve Eq. (4.11) by a WKB-type approach. The final result for the attraction factor due to a phase transition, $p(b_{\rm vac}, \cdots) \equiv (\varphi(+\infty) - \varphi_m)/(\varphi(-\infty) - \varphi_m)$, reads

$$p(b_{\rm vac}, \cdots) = 2^{3/2}\pi^{-1}\Gamma(5/4)a^{1/2}(6b_{\rm vac})^{-1/8}\exp(-I)\cos\theta \ , \tag{4.12}$$

where $I = (1/4)\int_{-\infty}^{+\infty}[1 - 3\lambda(p)]dp$, and where the angle θ depends on the two functions $\lambda(p)$ and $s_{\rm vac}(p)$. [In the approximation $\lambda(p) = 1/3$, one finds $\theta = \int_{+\infty}^{-\infty}[-s_{\rm vac}(p)]^{1/2}dp - \pi/8$]. In the case of the QCD phase transition, one has $\beta_{\rm vac} \simeq 42.36\kappa$ and $f_{\rm vac} = (34/3)/51.25 \simeq 0.2211$. If κ is of order unity, $b_{\rm vac} \simeq 9.37\kappa$ is probably large enough to render valid the WKB result (4.12). This yields an attraction factor $p_{QCD} \simeq 0.49a^{1/2}\kappa^{-1/8}\cos\theta$. In the case of the electroweak phase transition, rough estimates give $b_{\rm vac} \simeq (\lambda/4)\kappa$ where λ denotes the quartic self-coupling of the Higgs. Its seems therefore probable that $b_{\rm vac}^{\rm electroweak} \lesssim 1$, so that the electroweak transition has only a weak attracting effect on φ. We conclude that phase transitions seem to have only a modest effect on φ. It would be at present meaningless to refine the calculation of the effect on φ of the electroweak and QCD phase transitions [even

the order of the transitions is in doubt, not to mention the precise redshift dependence of $\lambda(p)$ and $s_{\rm vac}(p)$]. In fact, until one has some understanding of the cosmological constant problem, it does not make much sense to compute any gravitational effect linked to phase transitions. In the following we shall therefore neglect the effect of the phase transitions with respect to that of the nine mass thresholds discussed above.

V. EVOLUTION OF THE DILATON DURING THE MATTER-DOMINATED ERA

The matter content of the universe near the end of the radiation era and during the subsequent matter era can be described as the superposition of a relativistic gas ("radiation", i.e. photons and three neutrinos in the standard picture) and of a non-relativistic one ("matter"; made of particles of mass $m_m(\varphi)$). From Eqs. (3.5) the source terms for the cosmological evolution equations (3.2) read $\rho = \rho_r + \rho_m$, $P = P_r + P_m$, $\sigma = -\alpha_m(\varphi)(\rho_m - 3P_m)$ with $P_r = \rho_r/3$, $P_m \simeq 0$, and $\alpha_m(\varphi) \equiv \partial \ln m_m(\varphi)/\partial\varphi$. Either from the definition (3.5a) or from the separate energy balance equations discussed below Eq. (3.6), one deduces that, during the expansion, $\rho_r \propto a^{-4}$ while $\rho_m \propto m_m(\varphi) a^{-3}$. Finally, the evolution of φ with respect to the p-time $p \equiv \ln a + \text{const.}$ is given by the equation

$$\frac{2}{3-\varphi'^2}\varphi'' + [1-\lambda(p,\varphi)]\varphi' = -[1-3\lambda(p,\varphi)]\alpha_m(\varphi) \ , \tag{5.1}$$

with $3\lambda(p,\varphi) = [1 + C m_m(\varphi) e^p]^{-1}$, C being some constant. In the approximation where the radiation era has already attracted φ very near a minimum φ_m of $m_m(\varphi)$, we can consider that $m_m(\varphi) \simeq$ const. in $\lambda(p,\varphi)$. Choosing now the origin of p at the equivalence between radiation and matter [$\rho_r(p=0) = \rho_m(p=0)$], we get simply $\lambda(p) = 3^{-1}(1+e^p)^{-1}$. Neglecting φ'^2 in Eq. (5.1) and using the harmonic approximation (4.6), we find that φ satisfies a linear differential equation which can be rewritten as a hypergeometric equation. Denoting $x \equiv e^p \equiv a/a_{\rm equivalence}$ we have

$$x(x+1)\partial_x^2\varphi + \left(\frac{5}{2}x + 2\right)\partial_x\varphi + \frac{3}{2}\beta_m(\varphi - \varphi_m) = 0 \ . \tag{5.2}$$

The condition of regularity of φ when $x \to 0$, say $\varphi(x=0) = \varphi_{\rm rad}$ ($\varphi_{\rm rad}$ denoting the value of φ at the end of the radiation era, before the transition to the matter era around $p=0$), selects uniquely the solution of (5.2) to be $\varphi_m + (\varphi_{\rm rad} - \varphi_m) \times F[a,b,c;-x]$. Here $F[a,b,c;z]$ denotes the usual (Gauss) hypergeometric series. The values of the parameters are

$$a = \frac{3}{4} - i\omega \ , \quad b = \frac{3}{4} + i\omega \ , \quad c = 2 \ , \tag{5.3}$$

with $\omega \equiv \left[\frac{3}{2}\left(\beta_m - \frac{3}{8}\right)\right]^{1/2}$. In other words, the attraction factor of the matter era up to the present time, $F_m \equiv (\varphi_{\rm now} - \varphi_m)/(\varphi_{\rm rad} - \varphi_m)$, is given by

$$F_m = F[a,b,c;-Z_0] \ , \tag{5.4}$$

where $Z_0 \equiv e^{p_0} \equiv a_{\rm now}/a_{\rm equivalence}$ denotes the (Einstein frame) redshift separating us from the moment of equivalence between matter and radiation. As Z_0 is large (see Eq.(6.5a)

below), we can use the asymptotic behavior of the hypergeometric function (together with the properties of Euler's Γ function) to get more explicit forms for F_m. Whatever be the sign of $\beta_m - 3/8$ (i.e. in the two cases where ω is real or pure imaginary) one can write

$$F_m = 2^{1/2}\pi^{-1/2}2^{2i\omega}\Gamma_2 e^{-\frac{3}{4}p}e^{i\omega p} + (i\omega \leftrightarrow -i\omega) \ , \tag{5.5}$$

with $\Gamma_2 \equiv \Gamma(2i\omega)/\Gamma(2i\omega + 3/2)$. Actually, from the estimate (4.6b) we expect β_m to be (much) larger than $3/8$ (indeed, if $m_m \sim 1$ GeV, one would need κ to be smaller than 9.2×10^{-3} to make $\beta_m < 3/8$). In that case (ω real), one can compute the modulus of the complex number Γ_2 in terms of elementary functions to get

$$F_m = \left[\frac{\mathrm{cotanh}(2\pi\omega)}{\pi\omega(\omega^2 + 1/16)}\right]^{1/2} e^{-\frac{3}{4}p_0}\cos\theta_0 \ , \tag{5.6}$$

with $\theta_0 \equiv \omega p_0 + 2\omega \ln 2 + \mathrm{Arg}(\Gamma_2)$. As in the case of attraction by mass thresholds (when $\beta_A f_A^{\rm in} \gtrsim 1$), the attraction factor (5.6) is proportional to a cosine (when $\beta_m > 3/8$) because Eq. (5.1) describes a damped oscillation around the minimum φ_m of $\ln m_m(\varphi)$. When $\beta_m < 3/8$, φ slowly rolls down toward φ_m without oscillating (overdamped oscillator). [See also Ref. [13] in which the transition between radiation domination and matter domination was approximated — in the analytical formulas — as being a sharp one].

VI. OBSERVABLE CONSEQUENCES OF A COSMOLOGICAL RELAXED MASSLESS DILATON.

Sections 4 and 5 have exhibited several efficient mechanisms for driving the VEV of the dilaton toward a value where it decouples from matter. However, none of these mechanisms is a perfect attractor. The important question remains of giving quantitative estimates of the residual coupling strength of the dilaton at various cosmological epochs and of the corresponding observable effects.

The quantitative estimates of the efficiency of the cosmological attraction of the dilaton depend very much on the universality, or lack thereof, of the dilaton couplings. If the dilaton coupling functions $B_g(\Phi)$, $B_F(\Phi)$, $B_H(\Phi)$ (the latter representing the class of couplings to the fundamental Higgs sector, if it exists as such) are unrelated functions, one expects the mass functions (2.10) to have minima (if any) at different values of φ, say φ_m^A. For instance, the lepton and quark masses will involve B_H while hadron masses will all be proportional to $B_g^{-1/2}\exp[-8\pi^2 b_3^{-1}k_3 B_3]$ ($B_3 \equiv B_{SU(3)}$). In such a non-universal case, the various mass thresholds, and phase transitions, will not attract φ to the same value, but will tend to reshuffle each time the value of φ. In that case, the only efficient fixing of the value of φ would arise during the matter era, φ being attracted toward of minimum of $m_m(\varphi)$ where the label "m" represents the type of matter which dominates the present universe.

By contrast, one can consider the case where all the dilaton coupling functions coincide, $B_i(\Phi) = B(\Phi)$. This case of universal coupling of the dilaton to matter has a suggestive simplicity. It looks like a natural generalization of the universal $e^{-2\Phi}$ coupling arising at the string tree level. In the universal $B(\Phi)$ case, all the mass thresholds, as well as the QCD phase transition and the matter era, tend to attract φ to a common value, some maximum

φ_m of $B(\varphi)$.[5] In the universal case, the cosmological evolution is an extremely efficient way of pinning down the value of φ. Moreover, as φ is pinned down to an extremum of $B(\varphi)$, i.e. to a value where $\partial B(\varphi)/\partial\varphi$ and $\partial m_A(\varphi)/\partial\varphi$ vanish, one can say that the universal dilaton coupling case illustrates some "Principle of Least Coupling" in the sense that the universe is attracted to dilaton values extremizing the strengths of the interaction. It would be worth exploring whether imposing this universality provides a sensible way of selecting a preferred class of string models. In the following, we leave open the two possibilities, universal/non-universal, in our discussion of the observable consequences of our scenario.

The earliest observational information we have about cosmology concerns the primordial abundance of the light elements (mainly Helium 4, with traces of Deuterium, Helium 3 and Lithium 7). Let us discuss the production of Helium 4 as an example. In the standard scenario of homogeneous primordial nucleosynthesis, the abundance of Helium is mainly determined by the neutron/proton ratio at the temperature where the rate of interconversion $n \leftrightarrow p$ due to weak interactions becomes slower than the cosmological expansion rate (freeze-out) (see Ref. [19]). Neglecting the small additional effect of free neutron decay, one can write an approximate analytical formula for the primordial Helium abundance (by weight), Y, of the form, $Y = 2/(\exp(aX)+1)$ where a is a pure number of order unity and where X denotes the following dimensionless combination of coupling constants and masses

$$X \equiv g_2^{4/3}(1+3g_A^2)^{1/3}\left(\frac{m_n-m_p}{m_W}\right)\left(\frac{m_{\rm Planck}}{m_W}\right)^{1/3}\left(\frac{g_*}{10.75}\right)^{-1/6} . \tag{6.1}$$

Here g_2 denote the $SU(2)$ coupling constant, $g_A \simeq 1.26$ the axial/vector coupling of the nucleon, and g_* the effective number of relativistic degrees of freedom at freeze-out (retained here to allow easy comparisons between the effect of a change in g_* — e.g. an additional light neutrino — and the effects of changing, e.g., Newton's constant $G = m_{\rm Planck}^{-2}$, or Fermi's one $G_F = g_2^2/8m_W^2$). A remarkable fact about the combination X is that it is numerically of order unity thanks to a delicate compensation between large $\left[(m_{\rm Planck}/m_W)^{1/3} \simeq (1.52\times 10^{17})^{1/3}\right]$ and, small $[(m_n-m_p)/m_W \simeq 1.61\times 10^{-5}]$ factors. This fact prevents us from proposing an educated guess of the quantitative dependence of X on the bare dilaton coupling constants $B_F(\varphi)$, $B_g(\varphi)$,.... Even the sign of $\partial \ln X/\partial \ln B^{-1}$ (when $B_i(\varphi) = B(\varphi)$) is unclear. On the other hand one can estimate that $\partial Y/\partial \ln X \simeq -0.44$ both from the rough analytical formula for $Y(X)$ and from the numerical computations of the dependence of Y on the neutron half-life or on g_*. We can therefore write the value of the Helium abundance predicted by a scenario modified by the presence of a dilaton as

$$Y^{\rm dil}(\eta) = Y^{\rm GR}(\eta) - 0.22\frac{\partial \ln X}{\partial \ln B^{-1}}\kappa(\varphi_{\rm rad}-\varphi_m)^2 , \tag{6.2}$$

where we have reestablished the slight dependence of Y upon the baryon to photon ratio, η. In the standard, general relativistic scenario the dependence of the GR-predicted abundances

[5] Note that a primordial (inflationary type) phase transition — as well as the electroweak one, if the Higgs sector is fundamental — could instead attract φ to a minimum of $B(\varphi)$ through a transient vacuum energy $\propto B(\varphi)$.

on η is crucially used, together with the observed values of the light-element abundances, to set upper bounds on η, and thereby upper bounds of the ratio of the present total baryon mass density to the closure density, Ω_b. The standard conclusion being that baryons fail to close the universe by at least a factor five, $\Omega_b < 0.2$ [19]. Eq. (6.2) [to be completed by the corresponding dilaton-modified predictions for the other light elements] suggest that a dilatonic universe could naturally accomodate $\Omega_b = 1$ if the value $\varphi_{\rm rad}$ of φ at freeze-out (i.e. just after the electron mass threshold) differs by a small (but not too small) amount from the minimum φ_m [For instance, in the case of the Helium abundance, the dilaton correction term on the right-hand side of Eq. (6.2) should be approximately -0.03, and $\partial \ln X/\partial \ln B^{-1}$ should be positive]. It would be interesting to reexamine in full numerical detail primordial nucleosynthesis within the type of dilaton scenario considered here to assess whether it could naturally reconcile $\Omega_b = 1$ with the observed abundances of light elements. Let us only note here that the rather modest attraction toward φ_m which is probably needed in such a scenario seems more natural in the non-universal case. Indeed, in the universal $B(\varphi)$ case, all the nine mass thresholds compound their effect to drive φ very near some universal minimum φ_m. More precisely, $\varphi_{\rm rad} - \varphi_m = F_r \times (\varphi_{\rm in} - \varphi_m)$ where $\varphi_{\rm in}$ is the "initial" value of φ (meaning in this work, before the electroweak phase transition)and where the total attracting power of the radiation era is given by

$$F_r(\kappa) = \left\{ \prod_{A=1}^{9} m_{\pm}(\beta_A f_A^{\rm in}) \right\} \times \left\{ \prod_{i=2,3} p(\beta_i^{\rm vac} f_i^{\rm vac}) \right\} \ . \tag{6.3}$$

The values of β_A and $f_A^{\rm in}$ to be used in the attraction factors of each of the nine mass thresholds have been given above [remember that the $+(-)$ sign corresponds to fermions (bosons)]. The second factor in Eq. (6.3) corresponds to the effect of the two known phase transitions electroweak (2) and QCD (3). In view of the uncertainty in the calculation of the effect of phase transitions, and anyway of their expected modest contribution (see above), we shall neglect the attraction power of these phase transitions in the following. The small but non-zero value of $\varphi_{\rm rad} - \varphi_m = F_r(\kappa)\Delta\varphi$ [with $\Delta\varphi \equiv \varphi_{\rm in} - \varphi_m$] implies that all the gauge coupling constants squared, $g^2 \propto B^{-1}(\varphi)$, differed, at the end of radiation era, from their present values g_0^2 by a fractional amount

$$\frac{g^2_{\rm rad} - g_0^2}{g_0^2} \simeq \frac{1}{2}\kappa(\varphi_{\rm rad} - \varphi_m)^2 = \frac{1}{2}\kappa(F_r(\kappa)\Delta\varphi)^2 \tag{6.4}$$

[where we used the fact that $\varphi_0 - \varphi_m \ll \varphi_{\rm rad} - \varphi_m$ because of the matter era attraction]. As one a priori expects $\Delta\varphi \equiv \varphi_{\rm in} - \varphi_m$ to be of order unity, the function $\frac{1}{2}\kappa F_r^2(\kappa)$, which is plotted in Fig. 2, illustrates the remarkable efficiency (in the universal case) of the radiation era in pinning down the values of the physical coupling constants.

During the subsequent matter era, φ is (in the universal case) further driven toward φ_m by the factor $F_m(\kappa, Z_0)$, Eqs. (5.4)-(5.6). The numerical value of the matter-era attraction factor F_m is proportional to $Z_0^{-3/4}$ where $Z_0 \equiv e^{p_0}$ denotes the redshift separating us from the epoch of equivalence between matter and radiation. In the approximation $m_m(\varphi) \simeq$ const. introduced at the beginning of Sec. V, this redshift is given by [13]

$$Z_0 = \frac{\rho_0^{\rm matter}}{\rho_0^{\rm rad}} \simeq 13350\ \Omega_{75}\ , \tag{6.5a}$$

where

$$\Omega_{75} \equiv 8\pi \overline{G}\, \rho_0^{\rm matter}/[3(75 \text{ km s}^{-1}\text{Mpc}^{-1})^2] = \rho_0^{\rm matter}/1.0568 \times 10^{-29} \text{g cm}^{-3} \ . \tag{6.5b}$$

Under the hypothesis of a spatially flat universe ($K = 0$), generally assumed in this paper, Ω_{75} is linked to the present value of Hubble's "constant", H_0, by $\Omega_{75} = (H_0/75 \text{ km s}^{-1}\text{Mpc}^{-1})^2$. In that case, the observational limits $50 < H_0/1 \text{ km s}^{-1}\text{Mpc}^{-1} < 100$ imply $0.44 < \Omega_{75} < 1.78$. On the other hand, if one assumes that the universe is spatially hyperbolic ($K = -1$), one must modify the coefficients of the evolution equation (3.3) for φ by retaining the K-dependent terms. However, it was shown in [13] that as long as $\Omega_{75} > 0.05$ this modification of Eq. (3.3) has a small effect, and that the matter-era attraction factor of $K = -1$ universes is well approximated by the $K = 0$ formula (5.6), with Z_0 given by Eqs. (5.5a), (5.5b). The main difference is that now Ω_{75} is not related to H_0, and can be smaller than 0.44. In fact, present observational data are compatible with $\Omega_{75} \sim 0.1$.

Finally, the scenarios considered here predict that the present value of φ, say φ_0, differs from the minimum φ_m by $\varphi_0 - \varphi_m = F_t(\kappa, Z_0)\Delta\varphi$ where $\Delta\varphi \equiv \varphi_{\rm in} - \varphi_m$ and where the total attraction factor is

$$F_t(\kappa, Z_0) \equiv F_r(\kappa)\, F_m(\kappa, Z_0) \ . \tag{6.6}$$

There are three kinds of presently observable consequences of having φ_0 near, but different from, φ_m: (i) violations of the (weak) equivalence principle; (ii) modifications of relativistic gravity, and; (iii) slow changes of the coupling constants of physics, notably the fine-structure constant α and Newton's constant G.

To discuss the modifications of the gravitational sector, we can make use of the results of Ref. [20] on the relativistic gravitational interaction of condensed bodies in generic metrically-coupled tensor-scalar theories. Indeed, the action describing the classical interaction of massive particles of various species under the exchange of the $g_{\mu\nu}$ and φ fields is given by $S_{g,\varphi} + S_m[g, \varphi, x]$ where $S_{g,\varphi}$ is given by (2.7b) and S_m by Eq. (3.4). This action is identical to the one studied in Sec. 6 of [20]. We conclude that, at the Newtonian approximation, the interaction potential between particle A and particle B is $-G_{AB}m_Am_B/r_{AB}$ where $r_{AB} \equiv |\mathbf{x}_A - \mathbf{x}_B|$ and

$$G_{AB} = \overline{G}(1 + \alpha_A^{(0)}\alpha_B^{(0)}) \ . \tag{6.7}$$

Here $\overline{G}$ is the bare gravitational coupling constant entering the action (2.7b), and $\alpha_A^{(0)}$ is the present strength of the coupling of the dilaton to A-type particles, i.e. the value of (3.6) taken at the cosmologically determined VEV φ_0. [In diagrammatic language, the two terms on the right-hand side of Eq. (6.7) are, respectively, the one-graviton exchange contribution ($\overline{G}$) and the one-dilaton exchange one ($\overline{G}\alpha_A^{(0)}\alpha_B^{(0)}$)]. Two test masses, made respectively of A- and B-type particles, will fall in the gravitational field generated by an external mass m_E with accelerations a_A and a_B differing by

$$\left(\frac{\Delta a}{a}\right)_{AB} \equiv 2\,\frac{a_A - a_B}{a_A + a_B} = \frac{(\alpha_A^{(0)} - \alpha_B^{(0)})\alpha_E^{(0)}}{1 + \frac{1}{2}(\alpha_A^{(0)} + \alpha_B^{(0)})\alpha_E^{(0)}} \simeq (\alpha_A^{(0)} - \alpha_B^{(0)})\alpha_E^{(0)} \ . \tag{6.8}$$

All precision tests of the gravitational interaction used macroscopic bodies made of (neutral) atoms. Let the labels A, B,... denote some atoms. In the approximation where one neglects

m_u/m_N, m_d/m_N, m_e/m_N, α and $\alpha_{\rm weak}$, the mass of an atom is a pure (dilaton-independent) number times a QCD-determined mass scale, say $u_3(\varphi)$. In this approximation $\alpha_A(\varphi) = \partial \ln m_A/\partial\varphi$ is independent of the type of atom considered and is equal to $\alpha_3(\varphi) = \partial \ln u_3/\partial\varphi$. The dilaton dependence of u_3 is determined by Eq. (2.9). Choosing u_3 so that its present value $u_3(\varphi_0)$ is numerically equal to the atomic mass unit, $u = 931.49432$ MeV, we see from Eqs. (4.6) that

$$\alpha_A(\varphi_0) \simeq \alpha_3(\varphi_0) = \beta_3(\varphi_0 - \varphi_m) = \beta_3 F_t(\kappa, Z_0)\Delta\varphi \ , \tag{6.9}$$

with $\beta_3 = \kappa\partial \ln u_3/\partial \ln B^{-1} \simeq 40.82\,\kappa$.

In this approximation, the dilaton mimics a usual Jordan-Fierz (-Brans-Dicke) field, i.e. a scalar field coupled exactly to T^μ_μ. The main observational consequences of the body-independent coupling (6.9) are modifications of post-Newtonian relativistic effects, $O(v^2/c^2)$ beyond the Newtonian $1/R$ interaction (weak gravitational field case)[6]. The latter are measured by the two Eddington parameters $\gamma_{\rm Edd} - 1$ and $\beta_{\rm Edd} - 1$ (which vanish in general relativity). From [20] we see that in the approximation (6.9)

$$1 - \gamma_{\rm Edd} = 2\frac{\alpha_3^2}{1+\alpha_3^2} \simeq 2(\beta_3)^2(F_t(\kappa, Z_0)\Delta\varphi)^2 \ , \tag{6.10}$$

$$\beta_{\rm Edd} - 1 = \frac{1}{2}\frac{\beta_3\alpha_3^2}{(1+\alpha_3^2)^2} \simeq \frac{1}{2}(\beta_3)^3(F_t(\kappa, Z_0)\Delta\varphi)^2 \ . \tag{6.11}$$

Note also that the value of Newton's gravitational constant (in Einstein units) is $G_N = \overline{G}(1+\alpha_3^2)$.

Much more sensitive tests of the existence of dilaton couplings are obtained by looking at violations of the weak equivalence principle, i.e. at the body-dependence of $\alpha_A(\varphi_0)$ beyond the QCD approximation (6.9). To do this, we shall retain the leading m_u/m_N, m_d/m_N, m_e/m_N and α corrections to the mass of an atom. First, the mass of the nucleons have the form, $m_p = m_{N3} + b_u m_u + b_d m_d + C_p\alpha$, $m_n = m_{N3} + b_d m_u + b_u m_d + C_n\alpha$, where $m_{N3}(\simeq u_3)$ is the pure QCD approximation to the nucleon mass, and where b_u, b_d, C_p/u_3 and C_n/u_3 are pure numbers (in the approximation of negligible strange-quark content one has $b_u = \langle p|\overline{u}u|p\rangle/2m_N$, $b_d = \langle p|\overline{d}d|p\rangle/2m_N$) [22]. Second, the mass of an atom can be approximately decomposed as

$$m({\rm Atom}) = Zm_p + Nm_n + Zm_e + E_3^{\rm nucleus} + E_1^{\rm nucleus} \ ,$$

where Z is the atomic number and N the number of neutrons, and where $E_3^{\rm nucleus}$ denotes the strong-interaction contribution to the binding energy of the nucleus, and $E_1^{\rm nucleus}$ the Coulomb interaction energy of the nucleus.

In terms of the baryon number $B \equiv N + Z$, the neutron excess $D \equiv N - Z$, and the Coulomb energy term $E \equiv Z(Z-1)/(N+Z)^{1/3}$, the mass of an atom can be written as,

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[6] In view of the positiveness of β_3, the recent results of [21] show that the deviations from general relativity are further quenched in the strong-gravitational-field case of binary neutron star systems.

$$m(\mathrm{Atom}) = u_3 M_3 + \sigma' B + \delta' D + a_3 \alpha u_3 E \ , \tag{6.12}$$

where M_3 is a pure number ($= B+$ strong-interaction binding contribution) and where we have defined

$$\sigma' \equiv \sigma + \frac{1}{2} C_n \alpha + \frac{1}{2} C_p \alpha + \frac{1}{2} m_e \ , \quad \delta' \equiv \frac{1}{2}\delta + \frac{1}{2} C_n \alpha - \frac{1}{2} C_p \alpha - \frac{1}{2} m_e \ ,$$

with the usual definitions for $\sigma \equiv \frac{1}{2}(m_u + m_d)(b_u + b_d)$, $\delta \equiv (m_d - m_u)(b_u - b_d)$. [Note the factor $1/2$ in the first term of the definition of δ']. Finally, by differentiating the logarithm of (6.12) we get a more precise expression than (6.9) for the dilaton coupling strength

$$\alpha_A(\varphi_0) \simeq \alpha_3(\varphi_0) + \frac{\partial \widehat{\sigma}}{\partial \varphi_0} \left(\frac{B}{M}\right)_A + \frac{\partial \widehat{\delta}}{\delta \varphi_0} \left(\frac{D}{M}\right)_A + a_3 \frac{\partial \alpha}{\partial \varphi_0} \left(\frac{E}{M}\right)_A \ , \tag{6.13}$$

where we have introduced $\widehat{\sigma} \equiv \sigma'/u_3$, $\widehat{\delta} \equiv \delta'/u_3$ and approximated $M_3 \simeq M \equiv m(\mathrm{Atom})/u_3$ in the corrections terms.[7] Finally, from Eq. (6.8) we get an equivalence-principle violation of the form

$$\left(\frac{\Delta a}{a}\right)_{AB} = (\kappa F_t(\kappa, Z_0) \Delta\varphi)^2 \left[C_B \Delta\left(\frac{B}{M}\right) + C_D \Delta\left(\frac{D}{M}\right) + C_E \Delta\left(\frac{E}{M}\right)\right]_{AB} \ , \tag{6.14}$$

where $(\Delta X)_{AB} \equiv X_A - X_B$ and where $C_B = \lambda_{u_3} \partial\widehat{\sigma}/\partial \ln B^{-1}$, $C_D = \lambda_{u_3} \partial\widehat{\delta}/\partial \ln B^{-1}$, $C_E = \lambda_{u_3} \lambda_\alpha a_3 \alpha$, $\lambda_{u_3} \equiv \partial \ln u_3/\partial \ln B^{-1}$ and $\lambda_\alpha \equiv \partial \ln \alpha/\partial \ln B^{-1}$. Numerically, our usual estimate (2.9) gives $\lambda_{u_3} \simeq 40.82$, and the idea of unification of gauge couplings at the string scale gives $\lambda_\alpha \simeq 1$. [E.g. in the simplest $SU(5)$-type GUT the value of the fine structure constant at the QCD-confining energy scale u_3 — such that $\alpha_{\mathrm{strong}}(u_3) \simeq 1$ — is given by

$$\alpha(u_3)^{-1} = (22/7)\alpha_{\mathrm{GUT}}^{-1} - (10/21)\alpha_{\mathrm{strong}}(u_3)^{-1} \simeq (22/7)\alpha_{\mathrm{GUT}}^{-1} \propto B(\varphi)] \ .$$

We have also $a_3 \alpha = 0.717$ MeV$/u_3 = 0.770 \times 10^{-3}$ from the fit of atomic masses to the Bethe-Weizsäcker formula. We can therefore estimate the coefficient of the nuclear Coulomb energy term in Eq. (6.14) to be $C_E \simeq 3.14 \times 10^{-2}$. As for the other two coefficients, C_B and C_D, it is much less clear how to estimate them. From the experiment-derived values of $\sigma = 35 \pm 5$ MeV and $\delta = 2.05 \pm 0.30$ MeV, plus the theoretical estimates $C_p \alpha = 0.63$ MeV, $C_n \alpha = -0.13$ MeV [22], one can compute $\widehat{\sigma} = 3.8 \times 10^{-2}$ and $\widehat{\delta} = 4.2 \times 10^{-4}$. From the point of view of their dilaton dependence $\widehat{\sigma}$ and $\widehat{\delta}$ are the sum of four terms proportional to m_u/u_3, m_d/u_3, m_e/u_3 and α. It is impossible at present to reliably guess the dilaton-dependence of the mass ratios $m_{\mathrm{quark}}/m_{\mathrm{hadron}}$ and m_e/m_{hadron}. The numbers we would get for $\partial\widehat{\sigma}/\partial \ln B^{-1}$ would be very different were we to assume our usual exponential link to the string scale, or some other assumption. It seems however reasonable to estimate that

[7] Note that the assumption of a universal $B(\varphi)$ is crucial to ensure that all the terms in Eq.(6.13) have in common a very small factor $\varphi_0 - \varphi_m$. If, e.g., the mass of leptons (and/or α) depended on a different function of φ than the mass of hadrons, Eq.(6.13) would, at best, predict that the equivalence principle is violated at the (unacceptable) level $\sim (O(\alpha) + O(m_{\mathrm{lepton}}/m_{\mathrm{N}}))^2$, in the favourable case where the universe is assumed to be dominated by hadronic matter.

the order of magnitude of $\partial\widehat{\sigma}/\partial \ln B^{-1}$ and $\partial\widehat{\delta}/\partial \ln B^{-1}$ will be at most that given by the exponential assumption (2.11), and at least that obtained by differentiating only the fine-structure constant contributions to $\widehat{\sigma}$ and $\widehat{\delta}$. This yields corresponding rough upper and lower bounds for the coefficients of the B and D contributions: $1.1\times 10^{-2} \lesssim |C_B| \lesssim 7.5$, $1.7\times 10^{-2} \lesssim |C_D| \lesssim 8.2\times 10^{-2}$. If these upper bounds are correct, one can check that the last term in Eq. (6.14) will be numerically dominant for pairs (A,B) having a large difference in atomic number [Indeed, E/M is roughly proportional to $Z^{2/3}$]. The largest effect would arise in comparing Uranium (for which $E/M \simeq 5.7$) with Hydrogen (any light element would do nearly as well). For such a pair, Eq. (6.14) yields

$$(\Delta a/a)_{\rm max} = 0.18(\kappa F_t(\kappa, Z_0)\Delta\varphi)^2 \ . \tag{6.15}$$

The right-hand side of Eq. (6.15) is plotted in Fig. 3 as a function of κ (assuming $\Delta\varphi = 1$, and $\Omega_{75} = 1$). As one a priori expects κ to be of order unity, Fig. 3 shows that, within the scenario considered here (including universal dilaton couplings), the present tests of the equivalence principle (at the $10^{-11} - 10^{-12}$ level) do not put any significant constraints on the existence of a massless dilaton.

The situation is even worse if we consider tests of post-Newtonian gravity. Indeed, from Eqs. (6.10) and (6.15) we have the link

$$\left(\frac{\Delta a}{a}\right)_{\rm max} \simeq 5.4\times 10^{-5}(1-\gamma_{\rm Edd}) \tag{6.16}$$

showing that the present and planned levels of testing of post-Newtonian gravity, i.e. 10^{-3} and 10^{-7} at best for $\gamma_{\rm Edd}$, correspond, respectively, to equivalence-principle tests at the levels 5×10^{-8} and 5×10^{-12}. The other link $\beta_{\rm Edd} - 1 = \frac{1}{4}\beta_3(1-\gamma_{\rm Edd}) \simeq 10.2\kappa(1-\gamma_{\rm Edd})$ shows that $\beta_{\rm Edd}$ tests do not fare essentially better.

The last observational consequence of our scenario to discuss is the residual present variation of the coupling constants of physics. From $\partial \ln\alpha/\partial \ln B^{-1} \simeq 1$, with $\ln B^{-1} =$ const. $+ \frac{1}{2}\kappa(\varphi(t)-\varphi_m)^2$ and a present time dependence of φ given by Eq. (5.6) [in which the leading term is $\cos(\omega p + \text{const.})$], we deduce that

$$\left(\frac{\dot\alpha}{\alpha H}\right)_0 = -\kappa\left[\omega\tan\theta_0 + \frac{3}{4}\right](F_t(\kappa,Z_0)\Delta\varphi)^2 \ . \tag{6.17}$$

Using $\omega \equiv [\frac{3}{2}(\beta_m - \frac{3}{8})]^{1/2}$ with $\beta_m \sim 40.8\kappa$ (if the particles dominating the universe have a mass not very different from the GeV scale), we have the approximate link $(\dot\alpha/\alpha H)_0 \sim -43\kappa^{-1/2}\tan\theta_0(\Delta a/a)_{\rm max}$. This link shows again that equivalence principle tests are the most sensitive way of searching for possible dilaton couplings [In the foreseeable future, ultrastable cold-atom clocks might probe the level $\dot\alpha/\alpha \sim 10^{-16}{\rm yr}^{-1} \sim 10^{-6}H_0$ which corresponds to $\Delta a/a \sim 10^{-8}$].

Finally, the time variation of the gravitational coupling constant (measured in Einstein units) is

$$\left(\frac{\dot G}{GH}\right)_0 = -2\left[\omega\tan\theta_0 + \frac{3}{4}\right](\beta_3 F_t(\kappa,Z_0)\Delta\varphi)^2 \ . \tag{6.18}$$

In actual $\dot G$ experiments one is comparing an orbital frequency n (e.g. let us consider that of a planet around the Sun) to an atomic frequency ν. Taking into account the adiabatic

invariants of the orbital motion (angular momentum and eccentricity) and assuming an atomic clock based on the Bohr frequency $\propto m_e\alpha^2$, the directly measured quantity will be

$$\frac{\dot n}{n}-\frac{\dot\nu}{\nu}=2\frac{\dot G}{G}+2\frac{\dot m_s}{m_s}+3\frac{\dot m_p}{m_p}-\frac{\dot m_e}{m_e}-2\frac{\dot\alpha}{\alpha}\ , \tag{6.19}$$

where m_s, m_p denote the masses of the Sun and of the planet. [Contrary to Eq. (6.18), Eq. (6.19) is valid in any system of units]. Eq. (6.19) gives finally[8]

$$\frac{\dot n}{n}-\frac{\dot\nu}{\nu}=-H_0\left[\omega\tan\theta_0+\frac{3}{4}\right](F_t(\kappa,Z_0)\Delta\varphi)^2\left[4\beta_3^2+5\beta_3-\beta_e-2\kappa\right]\ , \tag{6.20}$$

in which the term coming from Eq. (6.18) dominates. In spite of the large factor $(\beta_3/\kappa)^2\simeq(40.8)^2$, the scaling of the prediction (6.20) with the Hubble rate H_0 makes it pale in comparison with equivalence principle tests. [On the other hand, this large factor renders $\dot G$ experiments competitive with $\dot\alpha$ ones].

VII. CONCLUSIONS

Einstein's starting point in constructing general relativity was the interpretation of the universality of free fall in terms of a universal coupling of matter to a common metric tensor $g_{\mu\nu}$. It has since been felt that such a universal metric coupling was the only theoretically natural way of explaining how the long-range fields participating in gravity[9] could satisfy the high-precision tests of the equivalence principle (now reaching the 10^{-12} level).

The present work suggests that a universal multiplicative coupling of a long-range scalar field Φ to all the other fields, $\mathcal{L}_{\rm tot}=B(\Phi)\mathcal{L}_0(g_{\mu\nu},\Phi,A_\mu,\psi,\ldots)$ with $B(\Phi)$ admitting a local maximum, though a priori entailing strong violations of the equivalence principle, provides another theoretically natural way of explaining why no violations have been seen at the 10^{-12} level. It maybe worthwhile to summarize in qualitative terms[10] the basic reasons why a massless dilaton is rendered nearly invisible during the cosmological evolution: (i) Each time, during the radiation era, the universe passes through a temperature $T\sim m_A$ the A-type particles and antiparticles become nonrelativistic before annihilating themselves and disappearing from the cosmic soup; this provides a source term for the dilaton proportional to the φ-gradient of $m_A(\varphi)$, which attracts φ toward a minimum φ_m^A of $m_A(\varphi)$; Eq. (4.6b) and Fig.1 suggest that each such attraction is moderately efficient, leaving φ nearer to φ_m^A by

[8]The link between $\dot\varphi_0$ and $\dot n/n$ is more involved if n is the orbital frequency of a binary neutron star system: see [23], which must be completed by taking into account the changes in the rest-masses of the stars, and the non-perturbative gravitational self energy effects [21].

[9]In Einstein's theory gravity is mediated by only one, spin 2, field; but in metrically-coupled tensor-scalar theories, gravity is mediated both by a spin 2 and a spin 0 field. In the latter case, the universal metric coupled to matter is a combination of the two pure-spin fields: $g_{\mu\nu}^{\rm univ}=A^2(\varphi)g_{\mu\nu}^*$.

[10]Simplified quantitative estimates are provided below.

a factor $\sim 1/3$. (ii) Under the assumption of universality of the dilaton coupling functions $B(\varphi)$, the minima of all the mass functions $m_A(\varphi)$ will coincide and the ~ 9 mass thresholds of the radiation era will compound their effects to attract very efficiently φ toward some common minimum φ_m. (iii) In the subsequent matter era, φ will be continuously attracted toward a minimum of the mass function $m_m(\varphi)$ corresponding to the (nonrelativistic) matter dominating the universe. [Under the same universality condition this minimum will be again φ_m.] The attraction factor due to the matter era is inversely proportional to the 3/4th power of the redshift $Z_0 \sim 1.3 \times 10^4$ separating us from the end of the radiation-dominated era. (iv) As a consequence of the very efficient total attraction toward φ_m, the present strength of the coupling of the dilaton to any type of matter $\alpha_A(\varphi)$, being proportional to the φ-gradient of $m_A(\varphi)$, is very small. The present deviations from general relativity in the interaction between two masses, m_A and m_B, are proportional to the product $\alpha_A \alpha_B$ and are therefore extremely small. (v) The equivalence principle tests are very sensitive, but they probe only differences $(\alpha_A - \alpha_B)\alpha_C$ which, because of the known universal features of QCD-generated masses, contain as supplementary small parameters either the ratio of the quark masses to the nucleon mass, or the fine-structure constant.

From a theoretical point of view, our work suggests a criterion for selecting a preferred class of string models: namely those where string-loop effects preserve the universal multiplicative coupling present at tree-level, with a dilaton-dependent function admitting a local maximum[11]. It will take, however, an improvement in our current understanding of supersymmetry breaking in string theory to see whether the universality required by the Least Coupling Principle is a viable option, providing a reasonable selection criterion for SUSY breaking mechanisms. It is to be noted that in this paper we had always in mind the coupling of the (four dimensional) dilaton which is such an intimate partner of the graviton that it seems reasonable to assume that it remains massless in the low-energy world[12]. However, the cosmological attractor mechanism described here could also apply to the other gauge-neutral scalar fields (moduli) present in string theory. Because of threshold effects, the gauge coupling function B_F acquires a non trivial dependence on the moduli fields [24]. Therefore, we have here a possible mechanism for fixing the moduli to values where they decouple from the other fields.

From an experimental point of view, our results provides a new incentive to improving the precision of equivalence principle tests (universality of free fall, constancy of the constants,...). Fig. 3 suggests, when assuming that the curvature of $B(\varphi)$ near its maximum is of order unity — say $0.1 < \kappa < 10$ —, to look for a present level of violation of the universality of free fall somewhere between 10^{-14} and 10^{-23}. Actually, one should not consider

---

[11]The existence of a local maximum in $B(\Phi)$ is necessary. For instance, in the case of the tree-level coupling function $\exp(-2\Phi)$, Φ would continuously roll toward $-\infty$ during the cosmological expansion, and worse, by Eq. (4.4), Φ would cause deviations from general relativity, including violations of the equivalence principle, of order unity or more.

[12]It is enticing to assume that the presently obscure mechanism ensuring the vanishing of the cosmological constant allows both 'gravitational' fields to remain long-ranged in the low-energy world.

the results plotted in Fig. 3 too seriously. On the one hand, even within the precise assumptions made in the text, the predicted maximal value of $\Delta a/a$ contains an unknown factor $\simeq \Omega_{75}^{-3/2}(\varphi_{in}-\varphi_m)^2$ which could be $\gtrsim 10$. On the other hand, our assumption (2.11) with $\mu_A = 1$ has entailed a specific phasing of the various oscillations undergone by φ during the radiation era, i.e. specific choices of where on the curves of Fig. 1 $(\varphi(p)-\varphi_m)/(\varphi(-\infty)-\varphi_m)$ ends up being when $p \to +\infty$. It is possible that the assumption (2.11) has overestimated the combined attraction power of the radiation era mass thresholds. A different estimate is obtained by multiplying the WKB approximations (4.10a) (all valid as soon as $\kappa \gtrsim 1$), assuming that all the oscillation angles $\theta_\pm^A$ are randomly distributed on the circle. Under the latter assumption it makes sense to compute a rms value of the radiation era attraction factor ($\langle \cos\theta_\pm^A \rangle_{rms} = 1/\sqrt{2}$). Neglecting as above the effect of the phase transitions one finds $[F_r(\kappa)]_{rms} = 1.87 \times 10^{-4} \times \kappa^{-9/4}$ for the radiation era attraction factor, and from Eq. (5.6) with $\omega \gg 1$, $\langle \cos\theta_0 \rangle_{rms} = 1/\sqrt{2}$ and $\Omega_{75} = 1$, $[F_m(\kappa)]_{rms} = 1.47 \times 10^{-5} \times \kappa^{-3/4}$ for the matter era attraction factor. This leads to a total attraction factor $[F_t(\kappa)]_{rms} = 2.75 \times 10^{-9} \times \kappa^{-3}$ and to the following analytical estimate of the rms value of the maximum value of the equivalence principle violation

$$(\Delta a/a)_{rms}^{\max} = 1.36 \times 10^{-18} \kappa^{-4} (\Delta\varphi)^2 \ . \tag{7.1}$$

The comparison of Fig. 3 with Eq. (7.1) (valid if $\kappa \gtrsim 1$ and the angles $\theta_\pm^A$ are randomly distributed) indicates that the phasing of the radiation era oscillations tends to be destructive. It is possible that alternative assumptions, different from (2.11), yields values of $(\Delta a/a)$ nearer to the rms analytical estimate (7.1). This would have the consequence that presently planned satellite tests of the equivalence principle [25] which aim at the level $\Delta a/a \sim 10^{-17}$, would probe a larger domain of values of κ, $\Delta\varphi$ and Ω_{75}.

In conclusion, high-precision tests of the equivalence principle can be viewed as windows on string-scale physics. Not only could they discover the dilaton, but, by fitting observed data to the expected composition dependence (6.14) of the equivalence principle violation, they could give access to the ratios C_B/C_E, C_D/C_E which are delicate probes of some of the presently most obscure aspects of particle physics: Higgs sector and unification of coupling constants.

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FIGURES

FIG. 1. The factor by which φ is attracted (when the early universe cools down through $T \sim m_A$) toward a minimum φ_m of the function $m_A(\varphi)$ is plotted as a function of $b_A = \beta_A f_A^{\rm in}$. The solid (dashed) line corresponds to A being a fermion (boson).

FIG. 2. The solid line represents $\log_{10}[(g^2_{\rm rad} - g_0^2)/g_0^2]$ as a function of $\log_{10}\kappa$, i.e. the fractional deviation (left over at the end of the radiation era) of the gauge coupling constants $g^2_{\rm rad} \propto B^{-1}(\varphi_{\rm rad})$ from their present values g_0^2, versus the curvature κ of the function $\ln B^{-1}(\varphi)$ near its minimum. The dashed line represents an analytical estimate (when $\kappa \gtrsim 1$) of that deviation, obtained by assuming that the phases θ of the WKB results Eq. (4.10a) are randomly distributed.

FIG. 3. The solid line represents $\log_{10}(\Delta a/a)_{\rm max}$ as a function of $\log_{10}\kappa$, i.e. the expected present level of violation of the equivalence principle (when comparing Uranium with a light element) as a function of the curvature κ of the (string-loop induced) function $\ln B^{-1}(\varphi)$ near a minimum φ_m. The dashed line represents an analytical estimate (when $\kappa \gtrsim 1$) of that violation obtained by assuming random phases θ in Eqs. (4.10a) and (5.6).

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Target Space Duality in String Theory

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Abstract

Target space duality and discrete symmetries in string theory are reviewed in different settings.

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1 Introduction, Summary, and Omissions

String theory (see for example [158]) assumes that the elementary particles are one dimensional extended objects rather than point like ones. String theory also comes equipped with a scale associated, nowadays, with the Planck scale (10^{-33} cm). The standard model describing the color and electro-weak interactions is based on the point particle notion and is successful at the Fermi scale of about 100 Gev (10^{-16} cm) which is 10^{-17} smaller than the Planck scale.

The correspondence principle requires thus that string theory when applied to these low energies resembles a point particle picture. In fact string theory has an interpretation in terms of point-like field theory whose spectrum consists of an infinity of particles, all except a finite number of which have a mass of the order of the Planck scale. Integrating out the massive modes leads to an effective theory of the light particles.

There exists another class of theories whose spectrum consists as well of an infinite tower of particles; this is the Kaluza-Klein type. In such a class of theories [182, 199, 12], gravity is essentially the sole basic interaction, and space-time is assumed to have, in addition to four macroscopic dimensions, extra microscopic dimensions, characterized by some small distance scale. The standard model is assumed to be a low-energy effective action of the light particles resulting from the purely gravitational higher dimensional system.

String theory can also be viewed in many cases as representing a space-time with extra dimensions. Nevertheless, the effective low-energy theory, emerging from string theory, turns out to be different from that resulting from a field theory with an infinite number of particles; it possesses many more symmetries.

String theory shows also differences when physics is probed at a scale much smaller than the Planck one. In fact, there are various hints that in string theory physics at a very small scale cannot be distinguished from physics at a large scale. A very striking example of that feature is that a string cannot tell if it is propagating on a space-time with one circular dimension of radius R (a dimensionless number) times the Planck scale or $1/R$ the Planck scale (see figure 1.A). The discrete symmetry apparent in the example is termed *target space duality*. Moreover, there are indications that string theory possesses an extremely large symmetry of that nature. A study of that symmetry is the subject of the review.

The term duality has been used time and again in physics, always for very noble causes. The duality between particle and wave was the precursor of quantum theory. In the Ising model for magnets it was found that a spin system at temperature T has identical properties as the same system at an inverse temperature, once an appropriate dictionary is used[4]. This property

[4]Notice that this temperature duality, outside the second-order phase-transition point, relates different *non-critical* systems. These systems are *not* scale invariant. The duality we discuss in this review, instead, relates critical systems.

and its generalizations have been found in many more statistical mechanical systems, some of them (such as lattice gauge theories) are of interest to particle physicists; it has led to the notion of a disorder variable. In the presence of monopoles, the electromagnetic interactions posses a duality allowing the exchange of the magnetic and the electric properties. This type of duality is very explicit in certain supersymmetric systems. String theory evolved out of the study of the dual description of two body interactions in terms of the direct channel (s-channel) scattering and exchange process (such as t-channel). Many features of these various types of duality appear as facets of the duality addressed in this review; none is identical to it.

The structure of the review is the following: Section 2 is of an introductory nature; the basic tools and concepts are introduced in a simple setting. In the other sections they are applied in more involved cases.

In section 2 we discuss the notion of a string background that is a an allowed space-time for string propagation[5]. There are restrictions on those space-times as they need to obey various constraints. In the early days, it was thought that as a consequence of such constraints the bosonic string could only propagate in 26 dimensions while a superstring could only propagate in 10 dimensions. Nowadays, one is aware of an infinite number of different solutions of these constraints, however, their complete classification is not known. It is also not known if all such solutions can be smoothly related or if they consist of disconnected islands (see figure 1.B).

In the same section we focus on a particular class of such solutions in which space-time is flat but has a global structure, i.e., several of the dimensions are compactified on tori. In that subspace there are transformations relating the various solutions. Some of these transformations turn out to be symmetries, that is they relate solutions which seem different in terms of the data which classifies them, but actually describe the same physics; the $R \to 1/R$ duality is one of them (see figure 1.C). We discuss the group structure of this symmetry of the toroidal compactifications in detail. Moreover, we study relations between this symmetry group and gauge symmetries. The large gauge group thus uncovered can be viewed as a broken version of an even larger symmetry, offering a glimpse at the underlying symmetries of string theory.

An attempt to implement this symmetry on the low-energy effective theory indeed provides an example of what this symmetry could be like. This symmetry needs to relate massless and massive modes, as in string theory it turns out that the massive modes cannot really be decoupled, and they eventually return. This is also worked out in section 2.

Having treated the case of flat compactifications and having introduced the key ideas, one ventures into less chartered territory. In section 3 one discusses duality in models which seem closer to phenomenology. The string is moving on allowed backgrounds which consist of a special class of compact dimensions called Calabi-Yau spaces; such spaces manifest themselves in space-time supersymmetric theories. In order to study such systems, it is useful to introduce

[5]Not all such backgrounds need to have a geometric description even classically.

the notion of spaces of an orbifold nature. These are spaces which, in particular, contain conical singularities. A compact dimension consisting a closed line including its two end points is the simplest example of this nature; it is the $\mathbf{Z}_2$ orbifold of a circle, constructed by identifying a point X on the circle with $-X$ (see figure 1.D).

Point particle quantum mechanics senses the singular nature of these backgrounds, however, string theory smoothens these singular effects, a very encouraging feature. In section 3, string theory and duality on such objects are described. Another surprise of string theory is that in some cases a string cannot distinguish if it is moving on an orbifold or on a circle (see figure 1.E).

The methods developed for orbifolds are applied in section 3 to discuss additional aspects of the duality invariant low-energy effective action, as well as some aspects of duality on Calabi-Yau manifolds. Moreover, in the same section one approaches phenomenology closer by studying the string quantum corrections to the tree approximation, and the impact of the duality symmetry on them.

In section 4, duality properties of curved space-times are discussed. For example, a string cannot distinguish on which of the backgrounds displayed in figure 1.F it moves. The infinite cigar in figure 1.F represents a Euclidean continuation of a two dimensional black hole. In order to identify the manifolds related by duality symmetry, another method is explained in section 4. While progress can be made even for curved backgrounds, some degree of symmetry is required from the manifold in order to apply the method. The role of these extra symmetries is discussed, and the method is then applied to various examples:

- Neutral and charged black objects (black holes, black p-branes) and other singular objects.
- Cosmological string backgrounds (expanding and contracting universes, in the presence of matter).

This method leads to some surprising consequences:

- The interchange of singularities with horizons, and the removal of singularities in string theory.
- A relation between neutral objects and charged objects.
- Topology change in string theory.

Moreover, it is shown in section 4 that a particularly interesting target space duality, termed *axial-vector duality*, is related to gauge symmetries.

Finally, in section 5, we discuss some interplay between the worldsheet and the target space. In string theory, one is very careful to distinguish between the worldsheet and the target space. In sections 2-4 a large discrete global symmetry emerged in many types of target spaces; this symmetry bears quite some resemblance to global discrete symmetries on the worldsheet (called "modular transformations"). In section 5 we discuss some mathematical facts which may suggest that the worldsheet and the target space are not necessarily disjoint.

This concludes the introduction and summary of the review.

In the following we briefly discuss issues that shall not be reviewed extensively. Our choice of not treating in depth these topics is due to the review dealing with aspects of duality either more elementary or more closely related to the authors past work.

Readers less familiar with the subject may prefer to go first through the bulk of the review before returning to the omissions:

- Mirror symmetry is a particular target space duality that interchanges two string backgrounds corresponding, to leading order in α', to compact Kähler manifolds with vanishing first Chern class. Such manifolds are called "Calabi-Yau" (CY). In compactifying the heterotic or supersymmetric strings to four dimensions one is typically interested in Calabi-Yau manifolds with complex dimension d smaller than or equal to three. When $d = 3$ mirror symmetry is particularly interesting for various reasons. Firstly, it is a non-trivial duality relating a pair of backgrounds with no continuous isometries (and vastly different from each other). Secondly, it relates the complex-structure deformations of one CY background to the Kähler-class deformations of its mirror partner. Therefore, this symmetry is useful, for instance, for calculating worldsheet instanton corrections to Yukawa couplings (simply by doing the calculation for the large radius limit of the mirror background).

  Mirror symmetry deserves its own review. Therefore, we mention it only briefly when we discuss a sub-class of dualities in CY backgrounds in section 3. This subject is discussed extensively in the literature, for example, in [159, 41], and in [289].

  For a complex torus, mirror symmetry is identical to what is called "factorized duality" in this review. We may therefore extend the notion of mirror symmetry to the case of curved background with toroidal isometries. The remarkable property that mirror symmetry relates backgrounds with different topologies remains true in these cases as well.

  Mirror transformation is just one element in the complete discrete symmetry group of CY backgrounds. Generalized duality in such moduli spaces was studied in the context of $N = 2$ Landau-Ginzburg (LG) theories. At the infra-red fixed point of the renormalization group flow, and under certain condition on the superpotential, $N = 2$ LG models are conjectured to be CFTs equivalent, in some sectors, to CY ground states [223, 276, 161, 214, 287]. By studying the deformation space of the superpotential, one is able to find a

subgroup of the complete set of discrete symmetries [58, 212, 150, 67].

The rest of the symmetries can be found by studying the monodromy group associated with the differential equation (of Picard-Fuchs type) which gives the periods of the three-form Ω in terms of the complex-structure deformation moduli [41, 213, 37, 53, 70]. We will briefly mention these techniques whenever discussing related topics.

- If one compactifies time as well, one finds new phenomena and new discrete (stringy) symmetries, relating different backgrounds. For toroidal backgrounds, one finds that there exist compactifications with large extended symmetries, typically associated with Lorentzian lattice algebras [149, 230]. Furthermore, it was shown that in those cases duality acts ergodically on the moduli space [230].

  The particular element inverting the time-like compactification radius β is referred to as "temperature duality." Indeed, as it is well known in field theory, the analytical continuation of a compact time yields a theory at finite temperature $2\pi T = 1/\beta$. This β duality was shown in ref. [5] to be an exact symmetry to all orders in string perturbation theory. This duality implies that two Hagedorn-like phase transitions (see, for instance [249, 201]) exist, one at "low" temperature ($\beta_1 = \sqrt{2} + 1$ for the heterotic string), and the other at "high" temperature ($\beta_2 = \sqrt{2} - 1$) [15, 203].

  The intermediate-temperature phase of the heterotic string was studied in ref. [11] by generalizing to non-zero temperature the construction of a duality-invariant effective action proposed in ref. [143, 144], and described in sections 2 and 3. In [11] it was claimed that the phase of the heterotic string above the Hagedorn temperature corresponds to a non-critical superstring in $7 + 1$ dimensions.

- The emphasis of this review is on target space duality for the bosonic and heterotic strings. Yet, there are some interesting dualities in other types of strings, which are less understood but well worth mentioning. These dualities typically relate different string theories.

  The first example is provided by the open-string duality. The open string does not have winding modes, so the usual form of duality, relating strings compactified on a circle of radius R with strings on a circle of radius $1/R$, cannot hold in its usual form. Nevertheless, it was claimed in the literature that some properties of the $R \to 1/R$ duality survive in this case as well if, in addition, one interchanges the open-string boundary conditions, for example, from Neumann to Dirichelet-like ones [69, 169, 156].

  The second example of non-standard duality is provided by duality between type II strings. In this case, there exists a map between the $R \to 0$ limit (in one compact-space direction) of the type IIA string and the $R \to \infty$ limit of the type IIB [69, 84]. Type IIA strings are mapped into type IIB since the duality changes the GSO projection in the Ramond sector [153]. Therefore, this duality bears some resemblance with a mirror transformation. This map gives an equivalence between the two string theories, however, the corresponding

discrete symmetry cannot be interpreted as spontaneously broken gauge symmetry, in the way discussed in section 2.

- A formulation of the worldsheet action of the bosonic string, in which target space duality is manifest, was proposed in ref. [270, 274]. This string action is based on two sets of coordinates: the usual ones (X) and the "dual" coordinates ($\tilde{X}$). In this way the $O(d,d,\mathbf{Z})$ duality is a manifest symmetry of the worldsheet action. The price to be paid is the loss of manifest 2-d covariance.

  A doubling of space-time degrees of freedom in the low-energy effective action (vielbeins, in this case) occurs also in ref. [263]. There, a manifestly $O(d,d,\mathbf{R})$ form of the target-space effective action was obtained, and $O(d,d,\mathbf{R})$ was realized linearly, again at the price of loosing manifest Lorentz invariance (this time in target space).

  Similarly, the doubling of the dilaton + axion degrees of freedom was advocated in [252] for the heterotic string. This doubling allows for the construction of a four-dimensional low-energy effective action manifestly invariant under $SL(2,\mathbf{R})$ transformations of the S field. The $SL(2,\mathbf{Z})$ subgroup of $SL(2,\mathbf{R})$ has been conjectured to be an exact symmetry of string theory in [115, 257]. This hypothetical exact symmetry, which includes in particular a strong-weak coupling duality, was used in [115] as a means of fixing non-perturbatively the VEV of the S field. Even here Lorentz invariance is non-manifest.

  Besides the loss of manifest Lorentz invariance, in two last formulations, it is not yet known how to extend the construction of these actions to a completely duality-invariant effective action of the heterotic string. The missing ingredient is the knowledge of how to include non-Abelian gauge fields. Only after coupling the theory to non-Abelian fields, are the continuous symmetries of the previous formulations reduced to discrete (duality) symmetries. This reduction from continuous $O(6,22,\mathbf{R})$ to discrete $O(6,22,\mathbf{Z})$ symmetries was observed in the context of effective actions in ref. [143, 144], and will be discussed in section 2.

- Target space dualities are also present in the space of topological backgrounds (see for example [284, 82]). For the space of twisted $N=2$ topological conformal field theories it was found that the discrete symmetries are extended to continuous ones [152]. Mirror symmetry in the space of such background was discussed in ref. [286]. The study of target space duality in topological backgrounds is useful, for example, in calculating Yukawa couplings [49, 151, 52, 286]. Moreover, a duality resembling a strong-weak coupling duality has been observed in the moduli space of a particular topological background [93].

- The $R \to 1/R$ duality is also present when the worldsheet is discretized as it happens in matrix models [262, 163]. This symmetry turns into the regular circle duality in the continuum limit, where one recovers a string moving in one compactified space dimension.

The one-loop partition function is proportional to $R + 1/R$ [29, 248, 206], and therefore, it is manifestly duality-invariant.

- In section 4 we discuss a general procedure for finding pairs of curved backgrounds related by duality. This procedure applies whenever the backgrounds possess Abelian symmetries. In refs. [36, 118, 116, 72, 124, 148, 4] this procedure was extended to the case of backgrounds possessing non-Abelian symmetries. Unlike the case of Abelian duality, here one does not know yet how to deal properly with global issues.

- Finally, let us notice that target space duality can be related to the well-known electric-magnetic duality of four-dimensional vector fields. In the context of field theory this duality was studied in a general setting in ref. [121]. More precisely, the target space duality acts on the kinetic term of the low-energy effective action as a particular discrete subgroup of the electric-magnetic duality. This phenomenon can be extended to other dimensions. Target space generalized dualities (as mirror symmetry) act on the kinetic term of the effective action as dualities among forms. This phenomenon further justifies the name of duality for the string target-space one since in several ways it is indistinguishable, from the low-energy point of view, from the old form-duality. Duality among forms was studied extensively in the literature, especially in the context of supersymmetric theories, see for instance [60, 78, 269, 77, 28, 74, 75, 188, 47, 215, 175, 126, 245, 79]. Quantum aspects of duality were studied, for instance, in [275, 76, 25, 91].

2 Duality and Discrete Symmetries of the Moduli Space of Toroidal Compactifications

2.1 Introduction: 2-d Conformal Field Theories as String Backgrounds

String theories use as building blocks Conformal Field Theories (CFT) (for a review, see for instance [158, 134]). In order to understand the symmetries of the string theories it is useful to study several properties of CFTs and translate them to their string theory counterparts.

Assume one is given a certain Lagrangian, L, which is an exact CFT defined on a certain two-dimensional manifold, Σ. The Lagrangian is exactly solvable, in particular, the Virasoro central charge, c, the allowed states and the corresponding operators and their operator product expansion (OPE) coefficients have been extracted. Given these data one wishes to investigate if there exist other CFTs with the same value of c in the neighborhood of L. A neighborhood is defined and constructed in the following manner. One considers the most general Lagrangian L'

$$L' = L + \sum_i g_i f_i(z, \bar{z}), \tag{2.1.1}$$

where $f_i(z, \bar{z})$ are the operators in the spectrum of the theory L and g_i are appropriate coupling constants. One searches perturbatively for those couplings, g_i, which one can add to L such that the modified L' is also a CFT.

The spectrum of operators associated with L can be divided, using the classification of statistical mechanics, into three groups:

a) Operators whose dimension is larger than 2, which are called "irrelevant operators." Each of them has the property that if one adds it to the Lagrangian with an appropriate coupling, g_i (which will have a negative mass dimension), and if one considers the flow of the initial coupling g_i, in the modified Lagrangian, L', one finds that g_i decreases to zero in the infrared limit (thus justifying its name).

b) Operators whose dimension is smaller than 2; these are termed "relevant operators." Under L', g_i (which now has positive mass dimensions) flows towards large values in the infrared limit. If the theory is unitary it will flow to a theory with a smaller value of c [290].

c) Operators whose dimension is exactly 2. These are termed "marginal operators." If L' differs from L just by the addition of these marginal operators, then L' does not break classical scale invariance explicitly, as the g_i are dimensionless in this case. However, the g_i may change under renormalization; this causes the marginal operators to actually subdivide into three classes:

c1) Marginal operators whose couplings g_i in the modified Lagrangian turn out to be infrared-free on the worldsheet. Such operators effectively belong to the class (a) of irrelevant operators. They differ from operators that are irrelevant at the classical level in that g_i decreases logarithmically and not power-like towards the infrared.

c2) Marginal operators whose couplings g_i under the modified Lagrangian are asymptotically free and thus increase logarithmically towards the infrared, causing an instability. This turns the operators into relevant ones.

c3) Marginal operators whose addition to L does maintain the dimension of the coupling g_i, thus promoting L to a family of CFTs. These operators are called "truly marginal," and it is them who form a basis for a neighborhood of conformal field theories.

The space of different conformal theories in the neighborhood of the theory L is spanned by the corresponding coupling constants g_i. The number, N, of independent operators $f_i(z,\bar{z})$, $(i = 1, \ldots, N)$, which can be added simultaneously to L, such that L' is a CFT, counts the local dimension of the space. In general this is a perturbative statement. In many cases the exact form of the truly marginal operator, which started out as being described by $f_i(z,\bar{z})$, would need to be modified so as to maintain L' as a CFT for a finite value of g_i.

The space of *all* CFTs connected to L by truly marginal deformations is called the "(connected) moduli space" of L. Locally around L the moduli space reduces to the neighborhood described above. The dimension N itself is a local concept as it may change for specific finite values of g_i.

Considering the local neighborhood, $\mathcal{M}$, it turns out that in some cases one can span the same space $\mathcal{M}$ by applying a certain continuous group $\mathcal{G}$ to L. Moreover, one finds that there also exists a subgroup $\mathcal{G}_d$ of $\mathcal{G}$ which acts like a symmetry of the physical theory. Upon acting with an element g of $\mathcal{G}$, a theory L_1 defined at one point in $\mathcal{M}$ is transformed into another theory L_2, corresponding to a different point in $\mathcal{M}$ (parameterized by different values of the couplings g_i). When $g \in \mathcal{G}_d$, these two theories are actually physically equivalent. This identity holds for all the orbits of $\mathcal{M}$ under $\mathcal{G}_d$. The groups $\mathcal{G}$ and $\mathcal{G}_d$ depend on the particular nature of $\mathcal{M}$. A general classification is not yet available; we will discuss several cases in this report.

Let us now return to the language of string theory. The couplings g_i correspond to allowed target space backgrounds in which the string may propagate. For a bosonic string, the couplings are usually collected into the background metric $\hat{G}_{ij}(X)$, the antisymmetric-tensor $\hat{B}_{ij}(X)$, and the dilaton $\hat{\Phi}(X)$. These couplings are in general X dependent, where X is a target space coordinate. A typical worldsheet action S is of the form

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_0^{2\pi} d\sigma \int d\tau \left[\sqrt{g}g^{\alpha\beta}\hat{G}_{ij}(X)\partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta}\hat{B}_{ij}(X)\partial_\alpha X^i \partial_\beta X^j \right. \\ &\quad \left. -\frac{\alpha'}{2}\sqrt{g}\hat{\Phi}(X)R^{(2)}\right], \end{aligned} \tag{2.1.2}$$

where $g_{\alpha\beta}$ is the worldsheet metric, $g \equiv \det g_{\alpha\beta}$, α' is proportional to the inverse string tension and $R^{(2)}$ is the worldsheet scalar curvature.

One may wish to recall that the ultimate purpose of the worldsheet action is to be used in order to calculate the S-matrix for scattering in some target space. The scattering quanta represent the fluctuations in some target theory around the classical values $\hat{G}_{ij}$, $\hat{B}_{ij}$ and $\hat{\Phi}$ defined by

$$\begin{aligned} \tilde{G}_{ij}(X) &= \hat{G}_{ij}(X) + h_{ij}(X), \\ \tilde{B}_{ij}(X) &= \hat{B}_{ij}(X) + b_{ij}(X), \\ \tilde{\Phi}(X) &= \hat{\Phi}(X) + \phi(X). \end{aligned} \tag{2.1.3}$$

The fluctuations h_{ij}, b_{ij} and ϕ describe the massless graviton, antisymmetric tensor and dilaton in the target space. The existence of truly marginal operators on the worldsheet corresponds to the existence of massless particles in the target space. Similarly, the target space tachyon corresponds to a relevant worldsheet operator, and massive states in the target space correspond to irrelevant operators from the CFT point of view. In string theory they are always dressed in such a way as to be vertex operators whose dimension is $(1,1)$. The notation (d_L, d_R) means that the left- (right-) handed conformal dimension of the operator is d_L (d_R). The right-handed and left-handed dimensions are defined to be the scaling weights under the worldsheet coordinate transformation: $\bar{z} \to \bar{\lambda}\bar{z}, \quad z \to \lambda z$.

In this section we discuss the groups $\mathcal{G}$ and $\mathcal{G}_d$ for the case in which the geometric picture consists of a string moving on a background where d dimensions are compactified, and $\hat{G}$, $\hat{B}$, $\hat{\Phi}$ are X independent. These are termed "flat compactifications" or "toroidal backgrounds." In section 2.2 we first describe the $d = 1$ case, namely, a circle compactification, and we study its one-loop partition function. In section 2.3 we present the higher-genus circle duality. The general case for the bosonic and the heterotic strings is presented in sections 2.4 and 2.5. The identification of the elements of $\mathcal{G}_d$ with target space dualities will allow us to interpret most of the latter as residual broken gauge symmetries. This is discussed in section 2.6. Finally, in sections 2.7 and 2.8 we present duality invariant effective field theories realizing this interpretation.

2.2 $R \to \alpha'/R$ Duality in the One-Loop Partition Function

The simplest example of duality transformations [194, 247], and duality symmetry, is provided by a single bosonic string coordinate compactified on a circle of radius R. We set the worldsheet metric in (2.1.2) into an orthonormal form: $g_{\alpha\beta}(\sigma,\tau) = \eta_{\alpha\beta}$. (For the moment we have suppressed any dependence of the worldsheet metric on the global structure of the worldsheet). The worldsheet action is

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \partial_\alpha X \partial^\alpha X. \tag{2.2.1}$$

The compactification is defined by the period identification

$$X \approx X + 2\pi R m, \tag{2.2.2}$$

where m is an arbitrary integer.

Since $X(\sigma,\tau)$ satisfies a free wave equation, it admits a decomposition in terms of left- and right-movers

$$X(\sigma,\tau) = X_R(\sigma-\tau) + X_L(\sigma+\tau). \tag{2.2.3}$$

These modes, upon the continuation $\tau \to -i\tau$, become, respectively, the analytic and anti-analytic string coordinates. They have the mode expansion

$$\begin{aligned} X_R(\sigma-\tau) &= x_R - \sqrt{\frac{\alpha'}{2}} p_R(\sigma-\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{l\neq 0} \frac{1}{l} \alpha_l e^{+il(\sigma-\tau)}, \\ X_L(\sigma+\tau) &= x_L + \sqrt{\frac{\alpha'}{2}} p_L(\sigma+\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{l\neq 0} \frac{1}{l} \tilde{\alpha}_l e^{-il(\sigma+\tau)}, \end{aligned} \tag{2.2.4}$$

where $x = x_L + x_R$, and the dimensionless quantities p_L, p_R read

$$\begin{aligned} p_R &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\alpha'}}{R} n - \frac{R}{\sqrt{\alpha'}} m\right), \\ p_L &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\alpha'}}{R} n + \frac{R}{\sqrt{\alpha'}} m\right). \end{aligned} \tag{2.2.5}$$

The canonically conjugate momentum of $X(\sigma)$ is

$$P(\sigma) = \frac{1}{2\pi\sqrt{2\alpha'}} \left[p_L + p_R + \sum_{l\neq 0} \alpha_l e^{+il(\sigma-\tau)} + \sum_{l\neq 0} \tilde{\alpha}_l e^{-il(\sigma+\tau)} \right]. \tag{2.2.6}$$

The total momentum $P = \int_0^{2\pi} d\sigma P(\sigma)$, canonically conjugate to $x_L + x_R$, is

$$P = \frac{1}{\sqrt{2\alpha'}}(p_L + p_R). \tag{2.2.7}$$

From eqs. (2.2.5, 2.2.6, 2.2.7) one derives the canonical commutation relations

$$[x_L, p_L] = [x_R, p_R] = i\sqrt{\frac{\alpha'}{2}}, \quad [\alpha_m, \alpha_n] = [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n,0}. \tag{2.2.8}$$

The (normal ordered) Hamiltonian reads

$$H = L_{0L} + L_{0R}, \tag{2.2.9}$$

where

$$\begin{aligned} L_{0R} &= \frac{1}{2} p_R^2 + \sum_{l=1}^{\infty} \alpha_{-l}\alpha_l, \\ L_{0L} &= \frac{1}{2} p_L^2 + \sum_{l=1}^{\infty} \tilde{\alpha}_{-l}\tilde{\alpha}_l. \end{aligned} \tag{2.2.10}$$

Here L_{0L} and L_{0R} are separately conserved, due to conformal invariance, and are the zero modes of, respectively, the left and right Virasoro operators [158].

Notice that L_{0L} and L_{0R} are invariant under the transformation

$$\frac{R}{\sqrt{\alpha'}} \to \frac{\sqrt{\alpha'}}{R}, \quad m \leftrightarrow n. \tag{2.2.11}$$

Under eq. (2.2.11) p_R transforms into $-p_R$, whereas p_L is invariant. The oscillators α_n, $\tilde{\alpha}_n$ also transform in a simple way under $R/\sqrt{\alpha'} \to \sqrt{\alpha'}/R$:

$$\alpha_n \to -\alpha_n, \quad \tilde{\alpha}_n \to \tilde{\alpha}_n \tag{2.2.12}$$

(so that (2.2.11) together with (2.2.12) implies $\dot{X}_R \to -\dot{X}_R,\ \dot{X}_L \to \dot{X}_L$).

Equations (2.2.11, 2.2.12) provide the simplest example of a *Target Space Duality Symmetry*. This is a *stringy* property that does not exist in field theory. The difference between a point-particle field theory and string theory may be understood by realizing that an extended object – a string moving on a circle – can wrap around it. Its winding number m, i.e. the number of times the string wraps around the circle, is an integer number. In order to wrap around the circle the string must be stretched, so its energy increases with the winding number and with the radius of the circle as m^2R^2. This contribution to the energy of the string is to be added to the field-theoretical one, equal to the square of the center-of-mass momentum, which is quantized in units of $1/R$, and thus proportional to n^2/R^2. The total energy of the string turns out to be invariant under the transformations (2.2.11, 2.2.12), which inverts the radius of the circle, and interchanges winding numbers with momenta.

The one-loop partition function of the compactified bosonic string is

$$Z = \int_\Gamma d^2\tau \hat{Z}(\tau,\bar{\tau}) \sum_{p_L,p_R} \mathrm{Tr}\, e^{i\pi\tau L_{0R} - i\pi\bar{\tau}L_{0L}}. \tag{2.2.13}$$

Here τ denotes the modular parameter describing conformally inequivalent tori, Γ is the fundamental region of the τ-plane under the action of the torus mapping class group $SL(2,\mathbf{Z})$, and $\hat{Z}(\tau,\bar{\tau})$ is the contribution of all coordinates, other than our compactified X, to the partition function. This partition function was first presented in [157]; for more details see e.g. ref. [158]. The trace in eq. (2.2.13) is taken over the Hilbert space spanned by the oscillators α_l, $\tilde{\alpha}_l$. The sum over p_L, p_R extends to all momenta of the form (2.2.5). Symmetry of the partition function under the target space duality (2.2.11) follows by noticing that the integers m and n of eq. (2.2.5) are dummy variables in (2.2.13).

So far, we have proved that target space duality is a symmetry of the one-loop partition function, that is of the (free) string *spectrum*. If symmetry (2.2.11) extends to the higher-genus contribution to the partition function as well, i.e. to the interacting string, we may claim that *a small compactification radius ($R/\sqrt{\alpha'} \ll 1$) is completely equivalent to a very large one*

($\sqrt{\alpha'}/R \ll 1$). This statement, pointing out to the existence of a minimal length in string theory, will be proven in the next section.

To make contact with the expression for the worldsheet action in eq. (2.1.2) it is useful to rescale the coordinate X to RX. The new coordinate is dimensionless and has periodicity 2π ($X \approx X + 2\pi m$). Now, the action in (2.2.1) reads as in eq. (2.1.2)

$$\frac{1}{4\pi\alpha'}\int_0^{2\pi} d\sigma \int d\tau \hat{G}_{11}\partial_\alpha X \partial^\alpha X, \tag{2.2.14}$$

with the *dimensionful* metric $\hat{G}_{11} = R^2$. Furthermore, since duality involves explicitly the parameter α' it is convenient to define a dimensionless metric $G_{11} = \hat{G}_{11}/\alpha'$. In the same way we redefine R to be the dimensionless radius, namely

$$\frac{R}{\sqrt{\alpha'}} \to R. \tag{2.2.15}$$

The new metric and radius transform in a simple way under duality:

$$G_{11} \to \frac{1}{G_{11}}, \quad R \to \frac{1}{R}. \tag{2.2.16}$$

2.3 Duality in the Higher Genus Partition Function

String theory involves a summation over worldsheets of all genera. For target space duality to be a symmetry of string theory (at least in perturbation theory) it should be a symmetry on each genus g. In this section we discuss this issue. We work with the dimensionless coordinate X whose periodicity is 2π, and with the dimensionless compactification radius R.

In order to study duality on higher-genus Riemann surfaces, it is convenient to revert from the Hamiltonian representation of the partition function, used in the previous section, to its functional integral representation. The partition function on a genus g Riemann surface M_g reads, upon analytical continuation $\tau \to -i\tau$ [158],

$$Z_g = \int [dg_{\alpha\beta} dX d\hat{X}] e^{-\frac{R^2}{4\pi}\int_{M_g} d\tau d\sigma \sqrt{g} g^{\alpha\beta}\partial_\alpha X \partial_\beta X - \int_{M_g} d\tau d\sigma \hat{L}}. \tag{2.3.1}$$

The integral is extended to all two dimensional metrics $g_{\alpha\beta}$ compatible with the topology of M_g. The string coordinates have been split into X, and "the rest" $\hat{X}$, with Lagrangian $\hat{L}$.

The compactification of X is defined by generalizing to a genus g surface the period identification (2.2.2)

$$\int_{a_i} dX = 2\pi m_i, \quad \int_{b_i} dX = 2\pi n_i \tag{2.3.2}$$

The a_i and b_i are the canonical homology cycles of the surface M_g; they are shown in figure 2.A. The label i takes the values $1, ..., g$ [101]. We denote with ω_i the standard basis of closed

holomorphic (1,0)-forms spanning $H_{(1,0)}(M_g, \mathbf{Z})$, and obeying [101, 8]

$$\int_{a_i} \omega_j = \delta_{ij}, \quad \int_{b_i} \omega_j = \tau_{ij} = \tau_{1\,ij} + i\tau_{2\,ij}. \tag{2.3.3}$$

The period matrix τ_{ij} is symmetric, of dimension $g \times g$, and it is a function of the $3g-3$ modular parameters of M_g [101, 8]. The most general field X satisfying eqs. (2.3.2) reads [8]

$$X(z,\bar{z}) = X(z) + \bar{X}(\bar{z}) + \overset{o}{X}(z,\bar{z}). \tag{2.3.4}$$

The field $\overset{o}{X}$ is an arbitrary scalar field *globally defined* on M_g, whereas $X(z)$ is a holomorphic solution of the classical equations of motion

$$X(z) = 2\pi \sum_{j=1}^{g} \int_{z_0}^{z} m_j \omega_j + n_i \tau_{ij} \omega_j. \tag{2.3.5}$$

The coefficients m_i and n_i are those of eq. (2.3.2) and the integral is performed on an arbitrary path joining a fixed reference point z_0 with z. Equation (2.3.5) is independent of the path because the forms ω_i are closed.

By a standard choice of the gauge fixing of the general coordinate and conformal invariances, and by choosing the appropriate two-dimensional metric $g_{\alpha\beta}$ [158, 8] the genus g partition function reads

$$Z_g = \int_{\mathcal{M}_g} dm D(m) R \sum_{m_i, n_i} e^{-\pi R^2 (\mathbf{m}+\mathbf{n}\tau)^t \tau_2^{-1} (\mathbf{m}+\mathbf{n}\tau)}. \tag{2.3.6}$$

The integral in this equation is performed over the moduli space $\mathcal{M}_g$ of M_g, and the sum extends to all integer m_i, n_i. We denoted by $D(m)$ all contributions to the partition function which are independent of R and m_i, n_i. Boldface characters denote g-dimensional row vectors. The factor R in front of the sum is due to the X zero mode.

The sum in Z_g can be rewritten in a more compact form by performing a Poisson resummation on $\mathbf{n}$ (see for example [258])

$$\sum_{m_i, n_i} e^{-\pi R^2 (\mathbf{m}+\mathbf{n}\tau)^t \tau_2^{-1} (\mathbf{m}+\mathbf{n}\tau)} = R^{-g} \det \tau_2 \sum_{(\mathbf{m},\mathbf{k}) \in \mathbf{Z}^{2g}} e^{i\pi (\mathbf{p}_R \tau \mathbf{p}_R - \mathbf{p}_L \bar{\tau} \mathbf{p}_L)}, \tag{2.3.7}$$

where

$$\begin{aligned} \mathbf{p}_R &= \frac{1}{\sqrt{2}} \left(\frac{1}{R} \mathbf{k} - R\mathbf{m} \right), \\ \mathbf{p}_L &= \frac{1}{\sqrt{2}} \left(\frac{1}{R} \mathbf{k} + R\mathbf{m} \right). \end{aligned} \tag{2.3.8}$$

By sending $R \to 1/R$ and $\mathbf{k} \leftrightarrow \mathbf{m}$, and noticing that, as in the previous section, the integers $\mathbf{k}$ and $\mathbf{m}$ are dummy variables, we find

$$Z_g\left(\frac{1}{R}\right) = R^{2g-2} Z_g(R). \tag{2.3.9}$$

The complete partition function of the string is

$$Z(\Phi, R) = \sum_{g=0}^{\infty} e^{(1-g)\Phi} Z_g(R), \tag{2.3.10}$$

and Φ is the (constant) dilaton (see eq. (2.1.2)). Equation (2.3.9) implies therefore [34, 136]

$$Z\left(\Phi + 2\log R, \frac{1}{R}\right) = Z(\Phi, R). \tag{2.3.11}$$

Equation (2.3.11) is the generalization to all genera of the result obtained in the previous section. In contrast to the one-loop case, here the dilaton VEV Φ appears explicitly. The equivalence of two seemingly different partition functions proven here is rather formal, since $Z(\Phi, R)$ is ill-defined, due to the tachyon divergence. This difficulty however is not present in the interesting case of the heterotic string, which is still duality invariant, as we shall prove later. The conclusion we reach thanks to eq. (2.3.11) is therefore meaningful, and it refines the statement made at the end of the previous section, namely: *a compactification of small radius is equivalent to one of large radius, provided we also change the dilaton VEV according to* $\Phi \to \Phi + 2\log R$.

2.4 $O(d, d, \mathbf{Z})$ Duality for d-Dimensional Toroidal Compactifications

The worldsheet action describing a toroidal background is (for a review, see [158])

$$\begin{aligned} S &= \frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \left[\sqrt{g} g^{\alpha\beta} G_{ij} \partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j \right. \\ &\left. -\frac{1}{2}\sqrt{g}\Phi R^{(2)}\right]. \end{aligned} \tag{2.4.1}$$

Here X^i are dimensionless coordinates whose periodicities are 2π,[6] namely

$$X^i \approx X^i + 2\pi m^i. \tag{2.4.2}$$

The metric and antisymmetric tensor G_{ij}, B_{ij}, $i, j = 1, ..., d$, are the dimensionless ones, defined as in the end of section 2.2. The simplest case where $d = 1$ was already discussed in sections 2.2, 2.3. In string theory the total Virasoro central charge should vanish. For the usual bosonic string the matter accounts for 26 units of c. The action above describes only the d compactified coordinates, and therefore, its central charge is $c = d$. In general, the model S (2.4.1) is tensored with some other CFT, for example, a string moving in a $(26 - d)$-dimensional flat Minkowski space, such that the total value of c is indeed 26. Till further notice we will concentrate only on the internal sector described by (2.4.1).

The number of independent truly marginal operators for a generic d-dimensional background is d^2. This is an exact result as the model is Gaussian. The d^2 marginal operators are composed of the $d(d+1)/2$ operators

$$\sqrt{g} g^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j, \tag{2.4.3}$$

[6] The geometrical data of the torus, in the presence of torsion, is encoded in G_{ij} and B_{ij}.

and the $d(d-1)/2$ operators

$$\epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j, \tag{2.4.4}$$

with the corresponding symmetric couplings G_{ij} and antisymmetric couplings B_{ij}, all in all d^2 couplings and operators. We disregard for a while the dilaton operator and coupling.

The d^2 couplings are the data describing the CFT; they are organized in the form (G, B). We will occasionally find it convenient to arrange the data in a matrix E whose symmetric part is G and whose antisymmetric part is B,

$$E = G + B. \tag{2.4.5}$$

We call E the "background matrix."

We first solve for the spectrum of the theory (2.4.1). This is done by performing a canonical quantization on the worldsheet. Since we are studying theories with no Weyl anomaly (as it happens for strings at criticality) we can set the worldsheet metric into a diagonal orthonormal form

$$g_{\alpha\beta}(\sigma, \tau) = \eta_{\alpha\beta}. \tag{2.4.6}$$

(For the moment we have suppressed any dependence of the worldsheet metric on the global structure of the worldsheet).

The canonical momentum P_i associated with X^i is given by

$$2\pi P_i = G_{ij}\dot{X}^j + B_{ij}X^{j\prime} = p_i + \text{oscillators}, \tag{2.4.7}$$

where p_i is the string center of mass momentum. Since the coordinates X^i have periodicities 2π, it follows that the momenta p_i is quantized in integer units

$$p_i = n_i. \tag{2.4.8}$$

The Hamiltonian and momentum constraints take the form:

$$\begin{aligned}
H &= L_{0L} + + L_{0R} \\
&= \frac{1}{4\pi}\int_0^{2\pi} d\sigma \left[(2\pi)^2(P_i G^{ij} P_j) + X^{i\prime}(G - BG^{-1}B)_{ij}X^{j\prime} + 4\pi X^{i\prime}B_{ik}G^{kj}P_j\right] \\
&= \frac{1}{4\pi}\int_0^{2\pi} d\sigma (P_L^2 + P_R^2), \\
P_{La} &= [2\pi P_i + (G-B)_{ij}X^{j\prime}]e_a^{*i}, \quad P_{Ra} = [2\pi P_i - (G+B)_{ij}X^{j\prime}]e_a^{*i}, \\
\int_0^{2\pi} d\sigma P X' &= L_{0L} - L_{0R} = 0
\end{aligned} \tag{2.4.9}$$

Here e and e^* are defined by

$$\sum_{a=1}^{d} e_i^a e_j^a = 2G_{ij}, \quad \sum_{a=1}^{d} e_i^a e_a^{*j} = \delta_i^j, \quad \sum_{a=1}^{d} e_a^{*i} e_a^{*j} = \frac{1}{2}(G^{-1})^{ij}. \tag{2.4.10}$$

The vectors e_i are a basis to the compactification lattice Λ^d, such that the target space d-torus is $T^d = \mathbf{R}^d/\pi\Lambda^d$; the vectors e^{*i} are a basis to the dual lattice Λ^{d*}. The indices a, b label an orthonormal basis to the target space.

We deduce from eqs. (2.4.9) that P_L and P_R, even in the presence of the backgrounds, still decouple and that they describe left- and right-movers, respectively. The model is indeed Gaussian for all flat data E; this is instrumental in obtaining exact results. Now, the Hamiltonian H consists of a zero mode part (for which we retain the notation H), and the part describing the oscillators. These oscillators – of each coordinate – transform similar to (2.2.12) under duality, in a way that does not change the spectrum (as shown at the end of this subsection). We thus first discuss only the zero mode part, whose Hamiltonian reads

$$\begin{aligned} H &= L_{0L} + L_{0R} = \frac{1}{2}(p_L^2 + p_R^2) = \\ &= \frac{1}{2}\left[n_i (G^{-1})^{ij} n_j + m^i (G - BG^{-1}B)_{ij} m^j + 2m^i B_{ik} (G^{-1})^{kj} n_j\right]. \end{aligned} \tag{2.4.11}$$

The integers n_i are the momentum eigenvalues defined in eq. (2.4.8) and m^i are the so called winding modes defined in eq (2.4.2). The zero-modes p_L, p_R are given by

$$\begin{aligned} p_R &= [n^t + m^t(B - G)]e^*, \\ p_L &= [n^t + m^t(B + G)]e^*. \end{aligned} \tag{2.4.12}$$

Here, for simplicity, we suppressed all vector and matrix indices for all quantities involved. Note that by eq. (2.4.7) one obtains that a state with a non-zero winding number but no explicit time dependence still carries momentum. This phenomenon was detected first in the context of 4-dimensional gauge theories [281]; it was shown that a monopole in the presence of a theta term obtains electric charge. This was studied in various four- and two-dimensional systems [45, 44]. The B field plays a role very similar to that of a theta term as we shall see.

The Hamiltonian can be defined for closed worldsheets with genus 0 or 1. Given the Hamiltonian and its spectrum, the next task is to identify the group generating the moduli space of the whole set of Lagrangians [236, 238], and the subgroup $\mathcal{G}_d$ which maintains the physics of the models [234, 81, 146, 260, 140]. The moduli space for toroidal compactifications is isomorphic to $O(d, d, \mathbf{R})/(O(d, \mathbf{R}) \times O(d, \mathbf{R}))$ [236, 238], where $O(d, d, \mathbf{R})$ is the non-compact orthogonal group in $d + d$ dimensions. A convenient way of representing the elements $g \in O(d, d, \mathbf{R})$ is

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.4.13}$$

where a, b, c, d are $d \times d$ matrices, and such that g preserves the form J

$$J = \begin{pmatrix} O & I \\ I & O \end{pmatrix}, \tag{2.4.14}$$

namely,

$$g^t J g = J \quad \Rightarrow \quad a^t c + c^t a = 0, \quad b^t d + d^t b = 0, \quad a^t d + c^t b = I. \tag{2.4.15}$$

(A useful consequence of the above condition is that if $g \in O(d,d,\mathbf{R})$, then $g^t \in O(d,d,\mathbf{R})$. This is proven as follows: begin with $g^t J g = J$, and take the inverse of both sides. Using $J^{-1} = J$, one finds $g^{-1} J (g^t)^{-1} = J$. Multiplying from the left by g and from the right by g^t one finds

$$J = g J g^t,$$

which shows that $g^t \in O(d,d,\mathbf{R})$).

The (d,d) Lorentzian momenta (p_{La}, p_{Rb}) in (2.4.12) form an even self-dual Lorentzian lattice $\Gamma^{(d,d)}$ [236], namely, the Lorentzian length is even

$$p_L^2 - p_R^2 = 2 m^i n_i \in 2\mathbf{Z}, \tag{2.4.16}$$

and the Lorentzian lattice is self-dual. It is known [258] that all even self-dual (d,d) Lorentzian lattices are related by $O(d,d,\mathbf{R})$ rotations. Obviously, any $O(d,d,\mathbf{R})$ rotation of $\Gamma^{(d,d)}$ gives back an even self-dual Lorentzian lattice. Moreover, to any $\Gamma^{(d,d)}$ corresponds a particular toroidal background. The *Euclidean lengths* $p_L^2 + p_R^2$ in two lattices related by an $O(d,d,\mathbf{R})$ rotation are generally different, and give rise to different spectra.

As (p_L, p_R) transform as vectors under $O(d,d,\mathbf{R})$, the Hamiltonian (2.4.11) is invariant under rotations by the maximal compact subgroup $O(d,\mathbf{R}) \times O(d,\mathbf{R})$. Therefore, these rotations do not change the zero-mode spectrum. Thus, the solution-generating group $\mathcal{G}$ is $O(d,d,\mathbf{R})$, and the moduli space is locally isomorphic to the coset manifold $O(d,d,\mathbf{R})/(O(d,\mathbf{R}) \times O(d,\mathbf{R}))$.

Recall that the momenta p_L and p_R are specified by the background matrix E. Therefore, the way the solution-generating group $\mathcal{G}$ acts on E is defined by its action on the vectors (p_L, p_R).

The spectrum is expressed by (p_L, p_R) or, equivalently, by the bilinear form defined by the $2d \times 2d$ matrix M [146]

$$M(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{2.4.17}$$

The spectrum of the system as obtained in eq. (2.4.11) is given by the scalar

$$H = \frac{1}{2}(p_L^2 + p_R^2) = \frac{1}{2} Z^t M Z, \quad Z = (m_a, n_b). \tag{2.4.18}$$

Here Z is the vector of integers counting the winding number and the number of momentum quanta, corresponding to (p_L, p_R). Under the $O(d,d,\mathbf{R})$ transformation g, given in eq. (2.4.13), the matrix M transforms as

$$M_g \equiv g M g^t. \tag{2.4.19}$$

To find the corresponding transformation of the background matrix E, it is convenient to first embed the $O(d,d,\mathbf{R})/(O(d,\mathbf{R})\times O(d,\mathbf{R}))$ moduli elements $E=G+B$ in $O(d,d,\mathbf{R})$. This is done by looking at the $O(d,d,\mathbf{R})$ element

$$g_{E=G+B}=\begin{pmatrix} e & B(e^t)^{-1} \\ 0 & (e^t)^{-1} \end{pmatrix}, \tag{2.4.20}$$

where the vielbein e is defined such that $G=ee^t$. Next, we define the action of $g\in O(d,d,\mathbf{R})$ (2.4.13) on a $d\times d$ matrix F by the fractional linear transformation

$$g(F)\equiv(aF+b)(cF+d)^{-1}. \tag{2.4.21}$$

It now follows that

$$g_E(I)=E=G+B. \tag{2.4.22}$$

Here I is the d-dimensional identity matrix. Moreover, one finds

$$M=g_E g_E^t. \tag{2.4.23}$$

From eqs. (2.4.19, 2.4.21, 2.4.22, 2.4.23) it follows that

$$M'=M(E')\equiv M_g=gMg^t=gg_Eg_E^tg^t=g_{E'}g_{E'}^t \quad\Rightarrow\quad g_{E'}=gg_E,$$

and as

$$E'=g_{E'}(I)=gg_E(I)=g(E),$$

one finally finds that the $O(d,d,\mathbf{R})$ transformation properties of the data E read [146, 260, 140, 141]:

$$E'\equiv g(E)=(aE+b)(cE+d)^{-1}. \tag{2.4.24}$$

(It is easy to verify that this action is consistent with the group property: $g(g'(E))=(gg')(E)$.)

Now that we know how $\mathcal{G}=O(d,d,\mathbf{R})$ sweeps the full moduli space of toroidal backgrounds, we may try to detect if in the process there still remains some symmetry group $\mathcal{G}_d$. A necessary condition for a group element to belong to $\mathcal{G}_d$ is that it leaves the spectrum of S invariant. It is allowed to change each state separately but the total spectrum should remain unchanged. The group element should act like an automorphism on the space of states. However, more is required in order to prove that both theories are equivalent: one ought to show that all correlation functions in one theory can be mapped into identical correlation functions in the other theory. In this section we will discuss only the above mentioned necessary condition on the spectrum, and in section 2.6 we will show that there exists a gauge symmetry relating the different points connected by the symmetry group $\mathcal{G}_d$. This will guarantee the equivalence of the theories.

Returning to the spectrum, there are two types of symmetry operators one can identify explicitly [146, 260]:

1. **Integer theta-parameter shift** $\Theta_{ij} \in \Theta(\mathbf{Z})$:

One can add to the antisymmetric matrix B_{ij} an antisymmetric integral matrix $\Theta_{ij}(\mathbf{Z})$ (that is a matrix composed of integer numbers) without changing the physics. In other words, one can act with the $O(d,d,\mathbf{R})$ elements of the type

$$g_\Theta = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix}, \tag{2.4.25}$$

where $\Theta_{ij} \in \mathbf{Z}$ and $\Theta_{ji} = -\Theta_{ij}$. Indeed this element not only belongs to $O(d,d,\mathbf{R})$ but to its discrete subgroup $O(d,d,\mathbf{Z})$, made of the elements g with integer entries.

In a way, the fact that $\Theta(\mathbf{Z})$ are symmetries of the spectrum is evident from the role of B as a theta parameter: the constant B-term in S is a total derivative, and therefore, gives only topological contributions. For an integer Θ-shift the action is changed by an integer multiple of 2π, and therefore, does not contribute to the path integral.

2. **Basis change** $A \in GL(d,\mathbf{Z})$:

One can conjugate E by $E' = AEA^t$, where A describes a basis change of the compactification lattice Λ. This can be done by acting on E as in eq. (2.4.24) with the $O(d,d,\mathbf{Z})$ elements

$$g_A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \tag{2.4.26}$$

where $A \in GL(d,\mathbf{Z})$. (For example, some of these operations result in permuting the space-time dimensions and thus they do not change the physics.)

Explicitly, the invariance of the spectrum can be shown by considering the new bilinear form

$$\begin{aligned} Z^t M' Z &= n^t (A^{-1})^t G^{-1} A^{-1} n - m^t A (G - BG^{-1}B) A^t m \\ &\quad -2n^t (A^{-1})^t (G^{-1}B) A^t m. \end{aligned} \tag{2.4.27}$$

The original form,

$$Z^t M Z = n^t G^{-1} n - m^t (G - BG^{-1}B) m - 2n^t G^{-1} B m, \tag{2.4.28}$$

is obtained by redefining $m' = A^t m$, $n' = A^{-1} n$, which is an allowed transformation.

In addition to $\Theta(\mathbf{Z})$ and $GL(d,\mathbf{Z})$ in (1),(2) there is a less obvious element of $O(d,d,\mathbf{Z})$ which is a symmetry of the spectrum [146, 260, 140, 141]:

3. **Factorized duality** D_i:

$$g_{D_i} = \begin{pmatrix} I - e_i & e_i \\ e_i & I - e_i \end{pmatrix}. \tag{2.4.29}$$

Here e_i is zero, except for the ii component which is 1, and I is a d-dimensional identity matrix. It can be shown straightforwardly that this transformation leaves the partition function invariant as well.

Factorized duality is a generalization of the $R \to 1/R$ circle duality in the X^i direction. Explicitly, D_i takes $R_i \to \frac{1}{R_i}$ if the d-dimensional background is a direct product of a R_i-radius circle and a $(d-1)$-dimensional background, leaving the latter unchanged.

It turns out that the elements in (1),(2) and (3) generate the discrete group $O(d,d,\mathbf{Z})$ [146, 260, 140, 141]. An additional symmetry is the worldsheet parity $\sigma \to -\sigma$ which acts on the background by $B \to -B$ [141]. It is not included in $O(d,d,\mathbf{Z})$ since it interchanges p_L with p_R, and thus flips the sign of the Lorentzian norm $p_R^2 - p_L^2$. This completes the identification of the symmetry group $\mathcal{G}_d$.

To discuss the transformation of the oscillators in (2.4.7) under $O(d,d,\mathbf{Z})$ we have to define the mode expansion of coordinates and conjugate momenta:

$$\begin{aligned} X^i(\sigma,\tau) &= x^i + m^i\sigma + \tau G^{ij}(p_j - B_{jk}m^k) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^i(E)e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^i(E)e^{-in(\tau+\sigma)}], \\ 2\pi P_i(\sigma,\tau) &= p_i + \frac{1}{\sqrt{2}}\sum_{n\neq 0}[E_{ij}^t\alpha_n^j(E)e^{-in(\tau-\sigma)} + E_{ij}\tilde{\alpha}_n^j(E)e^{-in(\tau+\sigma)}]. \end{aligned} \tag{2.4.30}$$

Let us recall that the periodicity of the coordinates x^i is 2π and both the winding number m and the oscillator number n are integers.

The commutation relations of $\alpha_n^i(E)$, $\tilde{\alpha}_n^i(E)$, x^i and p_i are found by expanding the equal-time canonical commutation relations

$$[X^i(\sigma,0), P_j(\sigma',0)] = i\delta_j^i\delta(\sigma-\sigma'). \tag{2.4.31}$$

The non-zero ones are

$$\begin{aligned} [x^i, p_j] &= i\delta_j^i, \\ [\alpha_n^i(E), \alpha_m^j(E)] &= [\tilde{\alpha}_n^i(E), \tilde{\alpha}_m^j(E)] = mG^{ij}\delta_{m+n,0}. \end{aligned} \tag{2.4.32}$$

Here the oscillators and their commutation relations are background-dependent since we fixed the periodicity of the coordinates to be 2π.

By inserting the mode expansion of X and P into eq. (2.4.9), and after integration and normal ordering, one finds that the complete Hamiltonian (zero modes plus the oscillator contribution) is:

$$H = \frac{1}{2}Z^tM(E)Z + N + \tilde{N}. \tag{2.4.33}$$

Z and M were defined in eqs. (2.4.17) and (2.4.18), and the number operators are

$$N = \sum_{n>0} \alpha^i_{-n}(E) G_{ij} \alpha^j_n(E), \quad \tilde{N} = \sum_{n>0} \tilde{\alpha}^i_{-n}(E) G_{ij} \tilde{\alpha}^j_n(E). \tag{2.4.34}$$

The transformation of the background matrix E under duality was given in eq. (2.4.24). A pair of useful relations between the original metric G and the dual metric G' is:

$$(d + cE)^t G'(d + cE) = G, \quad (d - cE^t)^t G'(d - cE^t) = G. \tag{2.4.35}$$

The first relation is derived by writing $G' = (E' + (E')^t)/2$, and using the expression for E' from eq. (2.4.24) to evaluate the left hand side. The second relation can be derived by first writing the analog of eq. (2.4.24) for $(E')^t$, and then expressing G' in terms of this $(E')^t$. Since duality is a symmetry of the string, it has to act as a canonical transformation on all the oscillators; that is, it has to preserve the commutation relations (2.4.31). This fact, together with the mode expansion (2.4.30) uniquely fixes the action of duality on the oscillators α^i_n, $\tilde{\alpha}^i_n$. With the help of the relations (2.4.35), one finds that the transformations of the oscillators read [204]

$$\alpha_n(E) \to (d - cE^t)^{-1} \alpha_n(E'), \quad \tilde{\alpha}_n(E) \to (d + cE)^{-1} \tilde{\alpha}_n(E'). \tag{2.4.36}$$

The number operators in the Hamiltonian (2.4.34) are manifestly invariant under (2.4.36), together with $G \to G'$ as given in (2.4.35), and therefore, we conclude that the entire spectrum is $O(d, d, \mathbf{Z})$-invariant.

Finally, under worldsheet parity ($B \to -B$) the left-handed oscillators are interchanged with the right-handed ones:

$$\text{worldsheet parity}: \quad \alpha_n \leftrightarrow \tilde{\alpha}_n. \tag{2.4.37}$$

The spectrum is manifestly invariant under this transformation.

Now, that we have specified how both the zero modes and oscillators transform under duality, one can deduce the transformation of states, $O|p_L, p_R\rangle$. (Here O is a polynomial in the oscillators, and $|p_L, p_R\rangle$ is the state corresponding to the operator $\exp(ip_L X_L(\bar{z}) + ip_R X_R(z))$.)

2.4.1 Generators of $O(d, d, \mathbf{Z})$ and $E \to 1/E$ Duality

We now discuss a particular element of $O(d, d, \mathbf{Z})$ which is an analog in d-dimensions to the $R \to 1/R$ circle duality. It is given by the *inversion* of the background matrix E [146, 260]

$$E = G + B \to E' = G' + B' = E^{-1}. \tag{2.4.38}$$

Here G' and B' are the symmetric and antisymmetric parts of E'. From eq. (2.4.38) it follows that the metric and the antisymmetric tensor transform as

$$\begin{aligned} G &\to G' = (G - BG^{-1}B)^{-1}, \quad B \to B' = (B - GB^{-1}G)^{-1}, \\ G^{-1}B &\to -BG^{-1}. \end{aligned} \tag{2.4.39}$$

Here G' (B') maintains the symmetry (antisymmetry) of G (B). The form for B' in this equation assumes that B is invertible (it is possible to write $B'(G,B)$ in a non-singular form even when B has zero eigenvalues).

Under the transformation (2.4.39) together with interchanging winding modes with momenta ($n \leftrightarrow m$) the Hamiltonian in (2.4.11) is manifestly invariant, and therefore, the spectrum as well. Moreover, from the expression for H in eqs. (2.4.9) it follows that $E \to 1/E$ interchanges $(2\pi\alpha')P \leftrightarrow X'$. Another way to argue that the interchange of P with X' is a symmetry is to notice that there exists a canonical transformation doing precisely that [146]. It is defined by the generating function of the old and new coordinates (the latter are denoted by a tilde):

$$\begin{aligned} F &= -\frac{1}{2\pi\alpha'}\int_0^{2\pi} X'(\sigma)\tilde{X}(\sigma)d\sigma, \\ P(\sigma) &= \delta F/\delta X(\sigma) = \frac{1}{2\pi\alpha'}\tilde{X}'(\sigma), \\ \tilde{P}(\sigma) &= -\delta F/\delta \tilde{X}(\sigma) = \frac{1}{2\pi\alpha'}X'(\sigma). \end{aligned} \tag{2.4.40}$$

Note that the canonical transformation (2.4.40) is equivalent to the simple change of backgrounds (2.4.38), (2.4.39) when these are constant.

In the absence of torsion ($B = 0$) one obtains the transformation

$$G \to G^{-1}. \tag{2.4.41}$$

Recall that, by our starting point (2.1.2):

$$G_{ij} = (\alpha')^{-1}\hat{G}_{ij}. \tag{2.4.42}$$

Here we have inserted back α', so $\hat{G}_{ij}$ is the conventional (dimensionful) metric and $(\alpha')^{1/2}$ is the string length parameter. We see that the transformation (2.4.41) corresponds to either $\alpha' \to (\det G)^2(\alpha')^{-1}$ at fixed G or, more conventionally, to $\hat{G} \to \alpha'\hat{G}^{-1}$ at fixed α'. Conventionally, one says that the theory is invariant under a change in $\hat{G}$ (at fixed α').

Duality can be given [146] a somewhat different meaning, which we illustrate in the case of $d = 1$, i.e. one compact dimension (discussed in section 2.2). Indeed, one may say that, given a string moving on a circle of radius R, the theory has to decide what is the best value of the "quantum" length $(\alpha')^{1/2}$. Such a choice has an intrinsic $\mathbf{Z}_2$ ambiguity given by eq. (2.4.41). If the value $\alpha' = R^2$ is (is not) picked up the "vacuum" does not (does) break spontaneously $\mathbf{Z}_2$.

It is known that, when the symmetric point is picked up, an enlarged gauge symmetry also follows corresponding to a level one $SU(2)_L \times SU(2)_R$ representation of the affine algebra (for a review, see for example [134]). Thus a $\mathbf{Z}_2$ preserving vacuum appears to preserve enlarged gauge symmetries as well.

The duality operator that inverts the background matrix E corresponds to the $O(d,d,\mathbf{Z})$ element

$$g_D = \begin{pmatrix} O & I \\ I & O \end{pmatrix}. \tag{2.4.43}$$

As explained before, the oscillators of X also transform under $O(d,d,\mathbf{Z})$ and, in particular, under the inversion duality g_D; in that case, their transformation is given by eq. (2.4.36) with $d=0$ and $c=I$. For example, at the self-dual point $E=I$, one finds $\alpha_n \to -\alpha_n$, $\tilde{\alpha}_n \to \tilde{\alpha}_n$.

We now consider some examples. In the $c=d=1$ case only the duality transformation g_D is available. The group $O(1,1,\mathbf{Z})$ consists of two elements[7], only one of which acts non-trivially on the compactification radius:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4.44}$$

The fractional linear transformation on G_{11} leads to

$$G'_{11} = \frac{1}{G_{11}}. \tag{2.4.45}$$

In this case, the manifold $\mathcal{M}$ defined in section 2.1 is shown in figure 1.C. The groups $\mathcal{G}$ and $\mathcal{G}_d$ defined there are $O(1,1,\mathbf{R})$ and $O(1,1,\mathbf{Z})$, respectively.

In the $c=d=2$ case, corresponding to two compactified dimensions, the factorized dualities, D_1 and D_2 (2.4.29) are represented by:

$$g_{D_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad g_{D_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2.4.46}$$

When the torus is a direct product of two circles ($G_{12}=B_{12}=0$) D_1 and D_2 correspond to $R_1 \to 1/R_1$ with R_2 fixed, and R_1 fixed while $R_2 \to 1/R_2$, respectively.

The generator Θ_{12} which causes the shift of the B_{12} field is represented by:

$$g_{\Theta_{12}} = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix}; \quad \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.4.47}$$

The group $GL(2,\mathbf{Z})$ is generated by three symmetries:

a) The permutation symmetry P_{12} giving rise to the element:

$$g_{P_{12}} = \begin{pmatrix} P_{12} & 0 \\ 0 & P_{12} \end{pmatrix}, \quad P_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4.48}$$

[7] Actually, $O(1,1,\mathbf{Z})$ has four elements, but minus the identity acts trivially as a fractional linear transformation. In general, whenever we discuss $O(d,d,\mathbf{Z})$ we actually mean the projective group $PO(d,d,\mathbf{Z})$.

b) The reflections on each space component i, R_i, given by:

$$\begin{aligned} g_{R_1} &= \begin{pmatrix} R_1 & 0 \\ 0 & R_1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\ g_{R_2} &= \begin{pmatrix} R_2 & 0 \\ 0 & R_2 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2.4.49}$$

Obviously, one can get R_2 from R_1 by a permutation: $R_2 = P_{12}R_1P_{12}$.

c) The transformation T_{12} of the form:

$$g_{T_{12}} = \begin{pmatrix} T_{12} & 0 \\ 0 & (T_{12}^t)^{-1} \end{pmatrix}, \quad T_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.4.50}$$

The generators P_{12}, T_{12}, R_1 generate $GL(2,\mathbf{Z})$. Its subgroup $SL(2,\mathbf{Z})$ is generated by T_{12} and the symmetry $S_{12} = R_1P_{12}$. The labels S and T are supposed to recall the standard $SL(2,\mathbf{Z})$ generators.

The previous examples are useful in analyzing the general case [141]. In the general $c = d$ case the group $O(d,d,\mathbf{Z})$ is generated by permutations P_{ij}, reflections R_i, the symmetries T_{ij} and Θ_{ij}, and the factorized dualities D_i, $i,j = 1,...,d$. In the $d = 2$ case the proof will be obtained in subsection 2.4.2 by referring to well known properties of the modular group $SL(2,\mathbf{Z})$; in the general case, this can be obtained [141] by straightforward, but rather tedious, algebraic manipulations.

Actually, it is enough to consider a single reflection and a single factorized duality, say R_1 and D_1, and a single transformation of the type T, and a single Θ, say T_{12} and Θ_{12}. The other elements can be obtained by repeated application of the previous ones, together with permutations, P_{ij}, $i,j = 1,...,d$.

We recall that we have shown the symmetry to occur for the partition function on the worldsheet torus. We will see in section 5 that it occurs for any worldsheet Riemann surface. The moduli space of allowed backgrounds has, therefore, a large degree of symmetry. String theories having a different description in terms of the conventional target space metric and torsion are actually physically identical. From a string theory point of view the target space description has fundamental domains. Moreover, these fundamental domains have an orbifold structure due to the occurrences of fixed points, that is there exist points in moduli space which are fixed under the symmetry group. It will turn out that these fixed points play an important role in the identification of the gauge symmetry inherent in the duality transformations. We thus turn to study some properties of these fixed points.

For the $c = 1$ compactification moduli space, as mentioned before, the fixed point in the moduli space of circle compactifications coincides with the appearance of enhanced gauge symmetries $SU(2)_L \times SU(2)_R$.

For the more general case, the duality transformation $E \to E^{-1}$ still has strictly speaking a single fixed point given by

$$G = I, \quad B = 0, \tag{2.4.51}$$

which is the unique solution to the equation $(G + B)^2 = I$ when G is positive definite. At the single fixed point the string forms a level-one representation of the affine algebra $SU(2)_L^d \times SU(2)_R^d$.

Let us consider now the slightly more general example of fixed points of inversion duality, $E \to E^{-1}$, modulo $SL(d, \mathbf{Z})$ and $\Theta(\mathbf{Z})$ transformations:

$$E^{-1} = M^t(E + \Theta)M, \quad M \in SL(d, \mathbf{Z}), \ \ \Theta \in \Theta(\mathbf{Z}). \tag{2.4.52}$$

It turns out that any background with a maximally enhanced symmetry falls into this category [146]. (By "maximal," we mean an enhanced semi-simple and simply-laced symmetry group of rank d corresponding to a level 1 affine Lie algebra). In those cases the background is [94]

$$E_{ij} = C_{ij}, \ \ i > j, \quad E_{ii} = \frac{1}{2}C_{ii}, \quad E_{ij} = 0, \ \ i < j, \tag{2.4.53}$$

where C_{ij} is the Cartan matrix. Thus,

$$E, \ E^{-1} \in SL(d, \mathbf{Z}), \tag{2.4.54}$$

and $M = E^{-1}$, $\Theta = E^t - E$ (i.e., $\Theta_{ij} = -C_{ij}, \ i > j, \ \Theta_{ij} = C_{ij}, \ i < j, \ \Theta_{ii} = 0$) solve eq. (2.4.52). This fact shows that a maximally enhanced symmetry point is a fixed point under some non-trivial $O(d, d, \mathbf{Z})$ transformation, and therefore, an orbifold point in the moduli space. Fixed points corresponding to non-maximal enhanced symmetries can be found by using factorized duality instead of the full inversion $E \to E^{-1}$.

We have not shown that the condition $E \in SL(d, \mathbf{Z})$ either implies an enhanced symmetry or exhausts all solutions. In fact, already for $B = 0$ one knows examples of models which correspond to physical fixed points, but do not have an enhanced affine algebra. Such systems are the 24-dimensional Leech lattice and the 8-dimensional E_8 lattice (with the full Cartan matrix as the background metric). These systems contain new $(n, 0)$ conserved currents, where $n > 1$.

Furthermore, examples where no higher spin conserved currents exist are also known, as described in the following subsection, where the special $c = d = 2$ case is studied in detail.

2.4.2 The $d = 2$ Example

The symmetry group, $\mathcal{G}_d$, of $d = 2$ flat compactifications is generated by $O(2, 2, \mathbf{Z})$ and $B \to -B$. As $O(2, 2, \mathbf{R})$ decomposes into $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ one may expect $O(2, 2, \mathbf{Z})$ to factorize into

$SL(2,\mathbf{Z}) \times SL(2,\mathbf{Z})$ [81, 260]. In fact, the moduli space is isomorphic to $SL(2,\mathbf{R})/U(1) \times SL(2,\mathbf{R})/U(1)$ but, instead, the duality symmetry group turns out to be isomorphic to $SL(2,\mathbf{Z}) \times SL(2,\mathbf{Z}) \otimes_S [\mathbf{Z}_2 \times \mathbf{Z}_2]$ [81, 260, 141].

To show this, one organizes the four real data, G_{11}, G_{12}, G_{22}, B_{12}, into two complex coordinates ρ and τ in the following manner [81]:

$$\begin{aligned} \tau &\equiv \tau_1 + i\tau_2 = \frac{G_{12}}{G_{22}} + \frac{i\sqrt{G}}{G_{22}} \\ \rho &\equiv \rho_1 + i\rho_2 = B_{12} + i\sqrt{G} \quad \text{where } G = G_{11}G_{22} - G_{12}^2 \end{aligned} \tag{2.4.55}$$

The inverse relation is:

$$E = \frac{\sqrt{G}}{\tau_2} \begin{pmatrix} \tau_1^2 + \tau_2^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.4.56}$$

where $b = B_{12}$. The target space is a two-dimensional torus and as such it supports a complex structure τ and a Kähler structure ρ. All in all, as we are going to explain, it turns out that there is a minimal set of four "generators" for the symmetry group $\mathcal{G}_d$. "Generators," in this context, means a set of group elements from which the whole group is generated by forming products.

A (non-minimal) set of generators is found as follows. $SL(2,\mathbf{Z})$ is generated by two elements, so a product of two $SL(2,\mathbf{Z})$ has four "generators." Two other generators are the one interchanging the two $SL(2,\mathbf{Z})$, and the reflection $X^1 \to -X^1$. Moreover, the worldsheet parity $\sigma \to -\sigma$ ($B \to -B$) provides a seventh generator. The $SL(2,\mathbf{Z})^2$ generators are

$$\begin{aligned} S: &\qquad \rho \to \rho, \quad \tau \to -\frac{1}{\tau}, \\ T: &\qquad \rho \to \rho, \quad \tau \to \tau + 1, \\ S': &\qquad \rho \to -\frac{1}{\rho}, \quad \tau \to \tau, \\ T': &\qquad \rho \to \rho + 1, \quad \tau \to \tau. \end{aligned} \tag{2.4.57}$$

The first two transformations reflect the fact that the target space is a 2-d torus, and thus unmodified by modular transformations of its complex structure.

The fourth transformation expresses the periodicity in the antisymmetric torsion B_{12}. The third transformation is a stringy symmetry that for $B = 0$ takes the volume in target space into its inverse.

Writing (p_L^2, p_R^2) in terms of τ and ρ as

$$\begin{aligned} p_L^2 &= \frac{1}{2\rho_2\tau_2} \left|(n_1 - \tau n_2) - \rho(m_2 + \tau m_1)\right|^2 \\ p_R^2 &= \frac{1}{2\rho_2\tau_2} \left|(n_1 - \tau n_2) - \bar{\rho}(m_2 + \tau m_1)\right|^2 \end{aligned} \tag{2.4.58}$$

exposes three more symmetries. The first one exchanges the complex and Kähler structures

$$D_2: \quad (\tau,\rho) \to (\rho,\tau). \tag{2.4.59}$$

This symmetry is of a stringy nature, it is actually the $O(2,2,\mathbf{Z})$ element we called *factorized duality* (2.4.29) (here acting on the second coordinate). This can be seen by setting $B_{12} = G_{12} = 0$. In this case eq. (2.4.59) corresponds to $G_{22} \to 1/G_{22}$. The other symmetry is

$$R: \qquad (\tau,\rho) \to (-\bar{\tau},-\bar{\rho}). \tag{2.4.60}$$

This equation corresponds to the $X^1 \to -X^1$ reflection we mentioned above. Indeed, this symmetry changes the sign of G_{12} and B_{12}, while leaving G_{11}, G_{22} invariant. The last symmetry, which corresponds to *worldsheet* parity ($\sigma \to -\sigma$), interchanges p_L and p_R and corresponds to $B \to -B$, while all other data are invariant. It is given by

$$W: \qquad (\tau,\rho) \to (\tau,-\bar{\rho}). \tag{2.4.61}$$

Since this symmetry interchanges p_L with p_R, it changes the sign of the Lorentzian norm $p_L^2 - p_R^2$ and thus cannot belong to $O(2,2,\mathbf{R})$.

The set of generators we presented is not a minimal one. One can remove the S', T' and R elements from the set of generators, as they can be obtained from S, T, D_2 and W by the help of eqs. (2.4.59, 2.4.61) ($S' = D_2 S D_2$, $T' = D_2 T D_2$, $R = D_2 W D_2 W$). The four symmetries (S, T, D_2, W) provide a minimal set of "generators" for the symmetry group.

The target space moduli space has the global structure of a product of the fundamental domains of two tori, that is $SL(2,\mathbf{R})/U(1) \times SL(2,\mathbf{R})/U(1)$, modded out by the discrete symmetries. In figure 2.B we show, as an example, the slice $\rho = \tau$ of the fundamental domain. A similarity between worldsheet and target space symmetries emerges. We will further discuss this issue later in section 5.

Not only do target space and worldsheet symmetries show similarity, there are cases where they could even be interchanged. Let us consider the one-loop partition function $Z(\tau,\rho;\sigma)$. It describes a string moving in the target space background defined by the data τ,ρ when the worldsheet is a torus with a given complex structure σ. This partition function is symmetric under the interchange of the complex structures of the target space and the worldsheet [81]

$$Z(\tau,\rho;\sigma) = Z(\sigma,\rho;\tau). \tag{2.4.62}$$

Together with the factorized duality, $\tau \leftrightarrow \rho$, this gives rise to a triality. This result also generalizes (partly) to higher dimensional backgrounds and higher genus worldsheets [140] as we discuss in section 5.

We conclude the example by noting several points in moduli space which are fixed points under some (non-trivial) duality.

Let us describe at first two points with a maximally enhanced worldsheet symmetry. The first corresponds to the $(SU(2)\times SU(2))_L\times(SU(2)\times SU(2))_R$ symmetric point, described by $E=I$, or equivalently by

$$(\tau,\rho)=(i,i). \tag{2.4.63}$$

The second corresponds to the $SU(3)_L\times SU(3)_R$ symmetric point, described by

$$E=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \tag{2.4.64}$$

or equivalently by

$$(\tau,\rho)=\left(\frac{1}{2}+\frac{i\sqrt{3}}{2},\frac{1}{2}+\frac{i\sqrt{3}}{2}\right). \tag{2.4.65}$$

Equations (2.4.63) and (2.4.65) are the two "orbifold singularities" in the fundamental domain shown in figure 2.B.

Finally, we present a continuous set of fixed points under the $E\to E^{-1}$ duality, combined with the $B\to -B$ worldsheet parity. These backgrounds correspond to the special value $\tau=i$, while $|\rho|=1$, or equivalently

$$E(g,b)=\begin{pmatrix}g & b\\ -b & g\end{pmatrix},\quad \text{s.t.}\quad g^2+b^2=1. \tag{2.4.66}$$

It now follows that

$$E(g,b)^{-1}=E(g,-b), \tag{2.4.67}$$

and therefore, it is a fixed point of WD (recall that $D=D_1D_2$). This continuous set of fixed points of an element of $\mathcal{G}_d$ includes two fixed points of $O(2,2,\mathbf{Z})$. The first is $b=0,g=1$, that is the symmetric background (2.4.63). The second is a fixed point of $T'D\in O(2,2,\mathbf{Z})$ at the special value $b=\pm 1/2, g=\sqrt{3}/2$. This last example shows [260] that there are orbifold points in the moduli space that do not have extra conserved chiral currents.

2.5 Duality of the Heterotic-String Spectrum

The Heterotic-string action in flat background, including the coupling to a background gauge field A_μ^α and a background antisymmetric tensor $B_{\mu\nu}$, $\mu,\nu=1,\ldots,d$, $\alpha=1,\ldots,16$ is (for a review, see [158])

$$\begin{aligned} S &= \frac{1}{2\pi}\int d^2z\left[(G_{\mu\nu}+B_{\mu\nu})\partial X^\mu\bar\partial X^\nu + A_{\mu\alpha}\partial X^\mu\bar\partial X^\alpha\right.\\ &\left. +(G_{\alpha\beta}+B_{\alpha\beta})\partial X^\alpha\bar\partial X^\beta\right]+(\text{fermionic terms}), \end{aligned} \tag{2.5.1}$$

with the constraint that the X_α are chiral bosons. Here we work in the orthonormal gauge (2.4.6), and the complex worldsheet coordinates and derivatives are

$$z=\frac{1}{\sqrt{2}}(\tau+i\sigma),\quad \partial=\frac{1}{\sqrt{2}}(\partial_\tau-i\partial_\sigma), \tag{2.5.2}$$

and therefore, $d^2z = d\sigma d\tau$, and $\partial x \bar{\partial} x = \frac{1}{2}(\partial_\tau x)^2 + \frac{1}{2}(\partial_\sigma x)^2$.

The index α denotes an internal index. The internal coordinates live on the weight lattice of $E_8 \times E_8$ or $Spin(32)/\mathbf{Z}_2$. The indices (μ, α) label a $(16+d)$-dimensional orthonormal basis: $G_{\mu\nu} \equiv \delta_{\mu\nu}$, $G_{\alpha\beta} \equiv \delta_{\alpha\beta}$.

The moduli space of the heterotic string in d-dimensional toroidal backgrounds is isomorphic, locally, to the symmetric space $O(d+16, d, \mathbf{R})/(O(d+16, \mathbf{R}) \times O(d, \mathbf{R}))$ [236, 238]. Here we will study the global structure of the moduli space following [146], namely, we will identify the target space discrete symmetries.

Due to chirality the geometrical interpretation of the heterotic background fields is not manifest. Therefore, we will "embed" the compactified section of the heterotic string (not including the worldsheet fermions) in $16+d$ space-coordinates of a bosonic string. The Lorentzian lattice of the bosonic string is of the form:

$$\begin{pmatrix} p_R \\ p_L \end{pmatrix} = \begin{pmatrix} p_R^i, & p_R^I \\ p_L^j, & p_L^J \end{pmatrix}, \quad i,j = 1,\ldots,d, \quad I,J = 1,\ldots,16. \tag{2.5.3}$$

The embedding is such that the lattice generated by

$$\begin{pmatrix} p_R^i & 0 \\ p_L^j & p_L^J \end{pmatrix} \tag{2.5.4}$$

is a sub-lattice of the original one, and gives the spectrum of the heterotic string. This will allow us to get the spectrum of the heterotic sting from that of the bosonic string by truncation. The sub-spectrum of the dual spectrum is given by a duality transformation of the original sub-spectrum. Thus, all one has to find is the E-matrix of parameters of the relevant bosonic coordinates.

Let $\{E_I^\alpha | I = 1,\ldots,16\}$, $\{e_i^\mu | i = 1,\ldots,d\}$ satisfy

$$\begin{aligned} E_I \cdot E_J = 2G_{IJ}, \qquad e_i \cdot E_I = 0, \\ e_i \cdot e_j = 2G_{ij}, \end{aligned} \tag{2.5.5}$$

where

$$\begin{aligned} G^{\alpha\beta} &= 2G_{IJ}(E^I{}^*)^\alpha (E^J{}^*)^\beta, \\ B^{\mu\nu} &= 2B_{IJ}(e^i{}^*)^\mu (e^j{}^*)^\nu, \qquad B^{\alpha\beta} = 2B_{IJ}(E^I{}^*)^\alpha (E^J{}^*)^\beta, \\ A^{\mu\alpha} &= 2A_{iJ}(e^i{}^*)^\mu (E^J{}^*)^\alpha, \end{aligned} \tag{2.5.6}$$

and $E^I{}^*$ $(e^i{}^*)$ is the basis dual to E_I (e_i).

The A field stands for an antisymmetric tensor and a metric in the coefficients E_{iI}, E_{Jj} of the background matrix. It is impossible for E_{IJ} to depend on A, as the gauge field is a

continuous parameter, while there are only discrete solutions to E_{IJ} which allow truncation of the spectrum of the $(16+d, 16+d)$ model to a chiral $(16+d, d)$ model.

The operator E of the embedding bosonic string has the form:

$$E \equiv B + G = \begin{pmatrix} \left(G + B + \frac{1}{4} A^K A_K\right)_{ij} & A_{iJ} \\ 0 & (G+B)_{IJ} \end{pmatrix} \tag{2.5.7}$$

To prove it, let us pick up a basis $\alpha \equiv (\alpha_i, \alpha_I)$ of a lattice $\Lambda^{(d+16)}$ with the metric $2G$:

$$\begin{aligned} \alpha_i &= (e_i, \frac{1}{2} A_i^K E_K), \\ \alpha_I &= (0, E_I). \end{aligned} \tag{2.5.8}$$

The dual basis α^* is

$$\begin{aligned} \alpha^{i\,*} &= (e^{i\,*}, 0), \\ \alpha^{I\,*} &= (-\frac{1}{2} A_i^I e^{i\,*}, E^{I\,*}). \end{aligned} \tag{2.5.9}$$

Thus, the bosonic Lorentzian lattice (p_L, p_R) has the form:

$$\begin{aligned} p_R &= [(n)^t + (m)^t (B-G)](\alpha^*) \\ &= n_i \alpha^{i\,*} + n_I \alpha^{I\,*} \\ &\quad + (m_i, m_I) \begin{pmatrix} \left(B - G - \frac{1}{4} A^K A_K\right)_{ij} & 0 \\ -A_{jI} & (B-G)_{IJ} \end{pmatrix} \begin{pmatrix} \alpha^{j\,*} \\ \alpha^{J\,*} \end{pmatrix} \\ &= ([n_i + m_j (B-G)_{ji} - \frac{1}{4} m_j A_j^K A_{Ki} - \frac{1}{2}(n_J + m_I (B+G)_{IJ}) A_i^J] e^{i\,*}, \\ &\quad [n_J + m_I (B-G)_{IJ}] E^{J\,*}). \end{aligned} \tag{2.5.10}$$

Here it turns out that $E_{IJ} = (G+B)_{IJ}$ is given by

$$E_{IJ} = C_{IJ}, \;\; I > J, \quad E_{II} = \frac{1}{2} C_{II}, \quad E_{IJ} = 0, \;\; I < J, \tag{2.5.11}$$

where C_{IJ} is the Cartan matrix of $E_8 \times E_8$. Therefore, the vectors V_J, W_J defined by

$$\begin{aligned} n_J + m_I (B+G)_{IJ} &\equiv V_J, \\ n_J + m_I (B-G)_{IJ} &\equiv W_J, \end{aligned} \tag{2.5.12}$$

are in the root lattice of $E_8 \times E_8$.

The truncation to a sub-lattice describing the heterotic string spectrum is the set of (n_I, m_I) such that $W_J = 0$. Defining P and L by

$$\begin{aligned} n_i e^{i\,*} &\equiv \frac{1}{2} P, \\ m_j (B-G)_{ji} e^{i\,*} &\equiv -L - BL, \end{aligned} \tag{2.5.13}$$

we get

$$p_R = (\frac{1}{2}P - L - BL - \frac{1}{2}VA - \frac{1}{4}A(AL), 0). \tag{2.5.14}$$

In the same way

$$p_L = [(n)^t + (m)^t(B+G)](\alpha^*) = (\frac{1}{2}P + L - BL - \frac{1}{2}AV - \frac{1}{4}A(AL), V + AL). \tag{2.5.15}$$

The vectors (p_L, p_R) span precisely the $(d+16, d)$-dimensional even self-dual Lorentzian lattice of the heterotic string [238]. Therefore, the bosonic truncated spectrum coincides with the heterotic model.

The A field stands for a metric

$$\begin{pmatrix} 0 & \frac{1}{2}A_{iJ} \\ \frac{1}{2}A_{Ij} & 0 \end{pmatrix}, \tag{2.5.16}$$

and an antisymmetric tensor

$$\begin{pmatrix} 0 & \frac{1}{2}A_{iJ} \\ -\frac{1}{2}A_{Ij} & 0 \end{pmatrix}, \tag{2.5.17}$$

in the cross directions. It also affects the metric in the compactified space: $G_{ij} \to G_{ij} + \frac{1}{4}A_i^K A_{Kj}$.

The background-inversion duality transformation for the bosonic string $E' = E^{-1}$, induces the transformation on the heterotic string background fields:

$$\begin{aligned} A'_{ij} &= -(G + B + \frac{1}{4}A^K A_K)^{-1}_{ij} A_{jI} (G+B)^{-1}_{IJ}, \\ (G' + B')_{ij} &= (G + B + \frac{1}{4}A^K A_K)^{-1}_{ij} - \frac{1}{4}A'^K_i A'_{Kj}, \\ (G' + B')_{IJ} &= (G+B)^{-1}_{IJ} \equiv (G+B)_{IJ} \mod SL(16, \mathbf{Z}) \text{ transformation}. \end{aligned} \tag{2.5.18}$$

So far we have described a particular element of the discrete target space symmetries of the heterotic string in flat backgrounds. Following the discussion for the bosonic string in section 2.4, one can show that here target space duality is generalized to a group isomorphic to $O(d+16, d, \mathbf{Z})$. This group is the subgroup of $O(d+16, d+16, \mathbf{Z})$ that preserves the heterotic structure of E in (2.5.7) (namely, that transforms the lower two blocks 0 and $(G+B)_{IJ}$ into themselves), while acting on E by fractional linear transformations (2.4.24).

2.6 Duality as a Gauge Symmetry

In this section we relate (most of) the discrete symmetry group $\mathcal{G}_d$ – the subgroup $O(d, d, \mathbf{Z})$ for d-tori compactifications – with a gauge symmetry.

2.6.1 Motivation

We have described in some detail the discrete symmetry group $\mathcal{G}_d$ which relates physically equivalent theories in the moduli space of flat compactifications; some of these symmetries are of a stringy nature, and are rather surprising from a field theory point of view. So far, it was shown that the worldsheet partition functions of any two theories related by a $O(d,d,\mathbf{Z})$ transformation are identical[8].

We will show that the theories themselves are *identical* by demonstrating that two theories related by $O(d,d,\mathbf{Z})$ are actually gauge equivalent under some gauge group. Uncovering such a gauge symmetry would afford one a glimpse into the structure of the large symmetry which is supposedly associated with string theory. A gauge symmetry could ensure that the symmetry group $O(d,d,\mathbf{Z})$ indeed persists to all orders in string perturbation theory. For example, such symmetries in $\mathcal{G}_d$ would be protected from explicit breaking had they actually been residual gauge symmetries surviving an incomplete gauge fixing of some continuous gauge group.

The discrete nature of the group $\mathcal{G}_d$ suggests that if it is a residual gauge symmetry it is to be associated with large gauge transformations, namely, gauge transformations which are not connected (in the space of gauge transformations) to the identity. Such a residual worldsheet gauge symmetry is familiar in string theory; it is the modular group acting on the worldsheet metric.

In string theory a large discrete symmetry, the modular group of Riemann surfaces (isomorphic to $SL(2,\mathbf{Z})$ on the torus, for example) is present; it reflects the underlying general coordinate invariance and Weyl invariance which are there when string theory is viewed as a theory of two dimensional gravity. This is the group of residual symmetries in the conformal gauge when the worldsheet is a torus. There are traces of general coordinate and Weyl invariance in the Veneziano and Virasoro-Shapiro amplitudes.

The similarity of the target space $\mathcal{G}_d$ symmetries with those of the worldsheet modular group encourages to view them as some gauged fixed version of a similar larger symmetry in the space of allowed string theory backgrounds.

Another hint in that direction emerges from the effective target space Lagrangian of the theory [84]. Consider the case of the bosonic string with one compactified dimension ($d=1$). The target space picture consists of an effective $SU(2)_L \times SU(2)_R$ scalar potential which has flat directions. The potential is constructed in terms of scalar fields in the (1,1) representation of the $SU(2)_L \times SU(2)_R$ symmetry group. A special point among all possible vacua is the one where all scalar fields obtain, say, a vanishing expectation value; in that case the $SU(2)_L \times SU(2)_R$ symmetry is retained. For any non-zero expectation values of the scalar fields, the symmetry is

[8] For $d>1$ we have shown explicitly only duality at genus $g=1$; the proof for $g>1$ is along the lines of section 2.3, and will also be discussed later in section 5.

spontaneously broken down to $U(1)_L \times U(1)_R$.

In the spontaneously broken case, the space of classical minima separates into gauge equivalent "shells;" each shell is a four dimensional surface characterized by its radius, and all points on a given shell are gauge equivalent (the surface is four dimensional as it is parametrized by the transformations associated with the four broken symmetry generators). In a finite scale invariant field theory (such as a four dimensional $N = 4$ SUSY theory), different surfaces would also be physically equivalent, as they represent equivalent theories which differ only by their scale; that scale is spontaneously generated and corresponds to the expectation value of a scalar field[9].

Using the $SU(2)_L \times SU(2)_R$ symmetry, any scalar expectation value in the adjoint representation can be pointed along a single component, for example, in the direction of the Cartan sub-algebra; let us denote the value of that component by h. A discrete gauge transformation, such as a rotation by an angle ϕ around an appropriate $SU(2)$ isospin axis (for example, if h points in the J^3 direction, a rotation around the J^1 axis can be used), rotates the scalar configuration h into $-h$. The two configurations are gauge equivalent (by a discrete gauge transformation). From the target space point of view, this identification corresponds to the $d = 1$ circle duality. Let us repeat that the two configurations are members of the same gauge equivalent surface, and therefore, are separated by a finite distance.

The correspondence of the above observation with target space duality is seen from the worldsheet perspective. The point where $SU(2)_L \times SU(2)_R$ is unbroken corresponds to the self-dual $R = 1$ point in the moduli space of circle compactifications. The spontaneous breaking of the target space symmetry (by giving an expectation value h to a target space scalar field) is achieved by an explicit breaking of the corresponding worldsheet symmetry [84]. The explicit breaking is done by adding to the worldsheet Lagrangian L_0, describing the self-dual point, a set of compatible marginal operators $f(z, \bar{z})$ to obtain the modified conformal Lagrangian, $L(\delta R)$, describing a string moving in one of the dimensions on a circle of radius $R = 1 + \delta R$,

$$L(\delta R) = L_0 + \delta R \, f(z, \bar{z}). \tag{2.6.1}$$

Expanding the duality relation $R' = 1/R$ around $R = 1$, one obtains the dual Lagrangian $L'(\delta R)$:

$$L'(\delta R) = L(-\delta R) + \mathcal{O}((\delta R)^2) = L_0 - \delta R \, f(z, \bar{z}) + \mathcal{O}((\delta R)^2). \tag{2.6.2}$$

This shows that the theories L and L', obtained by perturbing with $h \equiv \delta R$ or with $-h$, are dual to each other. But we have argued that they are also gauge equivalent by a transformation of

[9] In regular field theory this picture would be valid for all surfaces. However, in string theory things are more complex. The number of compactified dimensions, d, is bigger than 1, and at a finite distance from the point where the scalar fields had zero expectation value, the $SU(2)_L \times SU(2)_R$ symmetry is restored; the reappearance of massless states is special to string theory and will be dealt with in detail in section 2.8. We will ignore this extra structure for a while.

the target space $SU(2)_L$ (or $SU(2)_R$) gauge symmetry! Therefore, for $d = 1$, duality is related to a target space gauge symmetry.

The above consideration has another immediate worldsheet counterpart. The affine Lie algebra on the worldsheet is enhanced at the self-dual point from $U(1)_L \times U(1)_R$ to $SU(2)_L \times SU(2)_R$, with the appearance of two more (1,0) chiral currents, and two more (0,1) anti-chiral currents (the (1,0) and (0,1) indicate the scaling dimensions of these worldsheet operators). As will be explained in more detail soon, the perturbations h and $-h$ are related by a reflection of the current (or the anti-chiral current) contained in the marginal operator $f(z, \bar{z})$; this reflection is an element in the Weyl group of the enhanced symmetry $SU(2)_L$ (or $SU(2)_R$). Therefore, the two perturbations are related also by a worldsheet symmetry.

The relation between duality among theories near the self-dual point and Weyl symmetries [43] is valid all along the line of circle compactification (this will be shown for the general case). This concludes the discussion in the case of $d = 1$ flat compactifications.

As d increases, so does the complexity of the group $\mathcal{G}_d$. In particular, the couplings B_{ij} appear, and the moduli space has an increasing number of special points. Several questions arise: is there a generalization of the Weyl group construction of duality for general values of d? What is the relation between the symmetry group $\mathcal{G}_d$ and transformations appearing in the Weyl construction? Is there a preferred subgroup of special points to be utilized in the Weyl construction? What type of gauge group exists for general values of d? These issues are explored in the following subsection.

2.6.2 The Weyl Construction

The search for a basis of truly marginal operators in the moduli space, in the neighborhood of a CFT defined by some worldsheet Lagrangian, L, was discussed in section 2.1. For the purpose of the discussion in this section we will consider expansions around Lagrangians $L(E_0)$ defined at points with enhanced symmetry in the moduli space of d-dimensional toroidal compactifications (here E_0 denotes a background matrix corresponding to a point with enhanced symmetry). We do not know at this stage what is the general classification of such points. It may well be that all of them are needed to be considered in order to fully appreciate the larger gauge symmetry. We limit our discussion to those characterized by the existence of extra (1,0) currents and (0,1) currents, in addition to the Abelian ones. All those have been shown to be fixed points (under some elements of the duality group). It will turn out that they carry quite some information.

The moduli space of flat compactifications contains points with an enhanced affine symmetry group $G_L \times G_R$. We will discuss the discrete set of points with a *maximally enhanced symmetry*. By "maximally" we mean that $G_L = G_R = G$, where G is semi-simple and $\mathrm{rank}(G) = d$. (It is possible to generalize the study to any $G_L \times G_R$). These special points are described

by a background matrix, which we denote by E_M. The symmetric part of E_M, G_M, is one half the Cartan-matrix of a simply-laced group (modulo the discrete symmetry group). The antisymmetric part, B_M, is the Cartan valued torsion (2.4.53) (modulo the discrete symmetry group). The point E_M can be described by a level 1 WZNW model on the group manifold of G. Alternatively, the theory might be described in terms of d free bosons on the constant background E_M, with a Lagrangian $L(E_M)$. These theories contain the necessary extra (1,0) currents and (0,1) currents, in the adjoint representation of G_L and G_R.

Thus we are choosing to expand around special points where the spectrum of L is equipped with some current algebra. When the enhanced symmetry is $G_L \times G_R$, with $G_L = G_R = G$, the truly marginal operators can be composed out of the $(\dim G)^2$ (1,1) operators resulting from multiplication of the chiral and anti-chiral currents. The algebraic structure enables one to state a simple condition, valid to all orders in perturbation theory, on what are the allowed truly marginal operators.

Denote the chiral (anti-chiral) (1,0) ((0,1)) currents by J^a ($\bar{J}^b$), where $a, b = 1, ..., \dim G$. The perturbed theory

$$L(E_M, \epsilon) = L(E_M) + \Delta L, \qquad \Delta L = \epsilon_{ab} J^a \bar{J}^b, \tag{2.6.3}$$

is a truly marginal deformation of $L(E_M)$ whenever the matrix ϵ satisfies [56]

$$\sum_{n,m} \epsilon_{na} \epsilon_{mb} f^{nmc} = 0 \tag{2.6.4}$$

for all a, b, c. Here f^{nmk} are the structure constants of the group G. The perturbation ΔL in eq. (2.6.3) is equivalent to a perturbation

$$\Delta L' = \epsilon_{ab} J'^a \bar{J}'^b \tag{2.6.5}$$

if J' ($\bar{J}'$) are connected to J ($\bar{J}$) by a continuous transformation in the group G_L (G_R). Taking into account the "gauge" equivalence, the moduli space in the neighborhood of E_M is spanned by the (1,1) truly marginal operators in the Cartan sub-algebra of $G_L \times G_R$,

$$\Delta L = \epsilon_{ij} H^i \bar{H}^j, \qquad i, j = 1, ..., d, \qquad \text{rank}\, G = d. \tag{2.6.6}$$

Their number is $(\text{rank}\, G)^2$; a more involved way to appreciate this number is outlined below.

The number of truly marginal independent directions is $(\dim G)^2$ (generated by the $J^a \bar{J}^b$ operators, $a, b = 1, ..., \dim G$). But, because of the condition (2.6.4), the set of critical points that can be reached from $L(E_M)$ by conformal deformations spans a lower dimensional surface in the large space; its dimension is $2(\dim G - d) + d^2$. However, the dimension of the physical moduli space in the neighborhood of these points is only d^2. This is due to the fact that different truly marginal perturbations are equivalent under continuous transformations in the

group $G_L \times G_R$.[10] For example, in the $d = 1$ case, the space of critical points is a 5-dimensional surface – spanned by deformations of the type $(\sum_{a=1}^{3} \alpha_a J^a)(\sum_{b=1}^{3} \beta_b \bar{J}^b)$ (the overall scale is irrelevant) – in a 9-dimensional Euclidian space spanned by $J^a \bar{J}^b$, $a, b = 1, ..., 3$. The physical moduli space is one-dimensional, and generated, say, by the $J^3 \bar{J}^3$ operator.

Suppose one can relate ϵ' to ϵ, in the tangent space at the point E_M, by gauge transforming the currents H and $\bar{H}$. Then the discrete symmetry relating $L(E_M, \epsilon)$ to $L(E_M, \epsilon')$ is a spontaneously broken gauge symmetry of the enhanced symmetry group G. The symmetry is extended to the entire moduli space. The products of such transformations, originating at different enhanced symmetry points in the moduli space, correspond to a gauge transformation of a bigger group. We will discuss these points in more detail in what follows.

The symmetry group $\mathcal{G}_d$:

Let us recall briefly some properties of the discrete group $\mathcal{G}_d$ (see section 2.4). $\mathcal{G}_d$ is isomorphic to the semi-direct product of $O(d, d, \mathbf{Z})$ and a 2-dimensional parity symmetry. In the language of background fields, $2d$ parity relates the antisymmetric tensor B_{ij} to $-B_{ij}$. The subgroup $O(d, d, \mathbf{Z})$ acts on the background fields as a fractional linear transformation. The generators have been described in detail in subsection 2.4.1; here we only recall that one can generate $O(d, d, \mathbf{Z})$ by permutations P_{ij}, reflections R_i, the $SL(2, \mathbf{Z})$ symmetries T_{ij}, the antisymmetric tensor shifts Θ_{ij}, and the factorized dualities D_i, $i, j = 1, ..., d$.

Weyl symmetries as a part of $\mathcal{G}_d$:

We now turn to showing that the operations which are gauge transformations implemented by Weyl transformations, which themselves generate a group that we denote G_{Weyl}, can all be expressed as elements in $\mathcal{G}_d$. Therefore, many $\mathcal{G}_d$ elements will be shown to be gauge symmetries; we will also discuss the other $\mathcal{G}_d$ symmetries. In order to show this we need to prove a few statements:

Statement 1
Any element of G_{Weyl}, of a type expressing a pair of chiral and anti-chiral Weyl reflections, (W_L, W_R), around an enhanced symmetry point, E_M, corresponds to an element of $O(d, d, \mathbf{Z})$.

The corresponding $O(d, d, \mathbf{Z})$ transformation is denoted $g(W_L, W_R, E_M)$. We will expand on this statement below.

Statement 2
A general element in G_{Weyl} can be expressed as a product of Weyl reflections around enhanced

[10] The counting goes as follows: eliminating the gauge equivalence, by subtracting the $2(\dim G - d)$ broken gauge symmetries from the $[2(\dim G - d) + d^2]$-dimensional space, leaves d^2 physical directions.

symmetry points, all sharing the *same* symmetry.

Let us elaborate on this statement. There are many enhanced symmetry points (for $d > 1$ there are infinitely many points with enhanced symmetries for any group with rank d); nevertheless, the products of Weyl reflections around points with the same enhanced symmetry group (for instance, $SU(2)^d$) are sufficient to generate the full group G_{Weyl}. The content of the group itself does not depend on the special enhanced symmetry points used in its generation. The same is true if one forms products of Weyl reflections around points with several enhanced symmetry groups. This is due to the fact that an element of G_{Weyl} may be represented by many different products of reflections around different symmetry points; that is, the realization of G_{Weyl} in terms of reflections is not unique.

To describe the third statement, we first present some definitions. Let Λ be the lattice defining the torus of compactification, $T^d \simeq \mathbf{R}^d/\pi\Lambda$, of an enhanced symmetry point E_M, and let A be an automorphism of Λ, namely,

$$A(G_M)A^t = G_M, \tag{2.6.7}$$

where G_M is the symmetric part of E_M. Each Weyl reflection is, in particular, an automorphism of Λ, called *inner*. But there are automorphisms that are *not* Weyl reflections; these are called *outer* automorphisms. To an outer automorphism (for instance, a permutation of the two $SU(2)$'s in $SU(2) \times SU(2)$) corresponds an element $g(A, A, E_M)$ which is in the $GL(d, \mathbf{Z})$ subgroup ($A \in GL(d, \mathbf{Z})$) of $O(d, d, \mathbf{Z})$. The third statement is:

Statement 3
If $g \in O(d, d, \mathbf{Z})$ then $g = \prod_i g(A_{Li}, A_{Ri}, E_{Mi})$ for some enhanced symmetry points E_{Mi} .

Here (A_{Li}, A_{Ri}) is a pair of automorphisms, inner (Weyl) or outer, of the enhanced-symmetry root lattice Λ_i. The content of statement 3 is that *any* element of $O(d, d, \mathbf{Z})$ can be written as a product of the elements in G_{Weyl} (corresponding to Weyl reflections around points with enhanced symmetries) and *additional* elements relating current-current perturbations of enhanced symmetry points; the latter act on the currents as outer automorphisms (we elaborate on that below).

Statement 4

4a. The group $G_{Weyl}(d)$ is a subgroup of $O(d, d, \mathbf{Z})$, generated by the elements in $O(d, d, \mathbf{Z})$ that contain an even number of the generators $P_{ij}, T_{ij}, \Theta_{ij}$ (see their definition in subsection 2.4.1).

4b. The group $O(d, d, \mathbf{Z})$ is a subgroup of $G_{Weyl}(d + 2)$.

Here $G_{Weyl}(d)$ denotes the group G_{Weyl} for d-dimensional toroidal backgrounds.

The statement 4a tells us that the residual symmetries of the broken gauge group,

$$\prod_i g(W_{Li}, W_{Ri}, E_{Mi}), \tag{2.6.8}$$

(namely, the products of elements corresponding to pairs, (W_{Li}, W_{Ri}), of chiral and anti-chiral Weyl reflections around the enhanced symmetry points E_{Mi}), generate a special (infinite order) subgroup of $O(d,d,\mathbf{Z})$. Moreover, statement 4b tells us that *all* the elements of $O(d,d,\mathbf{Z})$ can be interpreted as spontaneously broken gauge symmetries in the group generated by products of Weyl reflections around points with enhanced symmetries in $d+2$ dimensions, $G_{Weyl}(d+2)$.

We are now going to elaborate on each of the above statements, and prove the various assertions.

The meaning of statement 1 is the following. In vector notation, the worldsheet Lagrangian $L(E_M, \epsilon)$ is defined to be

$$L(E_M, \epsilon) = L(E_M) + H^t \epsilon \bar{H}. \tag{2.6.9}$$

$L(E_M, \epsilon)$ is equivalent, physically, to the theory $L(E_M, \epsilon')$, where $\epsilon' = W_L \epsilon W_R^t$. This is so because $L(E_M)$ is unaffected by Weyl reflections acting on the (1,0) currents and (0,1) currents. The transformations

$$H' = W_L H, \qquad \bar{H}' = W_R^t \bar{H} \tag{2.6.10}$$

are equivalent to a change of the background matrix deformation ϵ to ϵ'. Now recall that ϵ and ϵ' are equivalent background perturbations of E_M if W_L and W_R are Weyl transformations. In that case the theories described by the background matrices $E_M + \epsilon$ and $E_M + W_L \epsilon W_R^t$ are connected by a gauge transformation in $G_L \times G_R$.

The particular element in $O(d,d,\mathbf{Z})$, $g(W_L, W_R, E_M)$, which corresponds to this gauge transformation, is found in the following manner. It is sufficient to find the $O(d,d,\mathbf{Z})$ transformation that coincides with the symmetry $\epsilon \to \epsilon'$ infinitesimally close to E_M. This is true because two isometries of a complete space, which coincide on the tangent space at the point p, and which preserve p, coincide on the entire space. A convenient metric on the moduli space is the Zamolodchikov's metric [290] (which is a metric of $O(d,d,\mathbf{R})/O(d,\mathbf{R})^2$ viewed as a symmetric space),

$$ds^2 = g_{ab}\, dx^a dx^b = Tr\left(G^{-1}\epsilon^t G^{-1}\epsilon\right). \tag{2.6.11}$$

Any automorphism satisfying eq. (2.6.7) is obviously an isometry acting on ϵ in the neighborhood of E_M. The discrete symmetry corresponding to the action of W_L and W_R is, therefore, an isometry (as it should be by definition).

We will look for an isometry in $O(d,d,\mathbf{R})$ which coincides with the action of the inner automorphisms W_L, W_R on the tangent space of E_M. Thus, we look for an element (see section 2.4)

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d,\mathbf{R}), \qquad g^t J g = J, \qquad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{2.6.12}$$

such that

$$g(E_M+\epsilon)\equiv[a(E_M+\epsilon)+b][c(E_M+\epsilon)+d]^{-1}=E+\epsilon'+\mathcal{O}(\epsilon^2). \tag{2.6.13}$$

Solving for ϵ' one finds

$$\epsilon'=(a-E_Mc)\epsilon(cE_M+d)^{-1}, \tag{2.6.14}$$

and we demand that $\epsilon'=W_L\epsilon W_R^t$, thus

$$a-E_Mc=W_L,\qquad cE_M+d=(W_R^t)^{-1}. \tag{2.6.15}$$

Finally, after some algebra one finds [43, 141]

$$g(W_L,W_R,E_M)=A(E_M)\begin{pmatrix}(W_R^t)^{-1} & 0\\ 0 & W_L\end{pmatrix}A^{-1}(E_M), \tag{2.6.16}$$

where

$$A(E)=\begin{pmatrix}E/2 & -E^tG^{-1}\\ I/2 & G^{-1}\end{pmatrix},\qquad A^{-1}(E)=\begin{pmatrix}G^{-1} & G^{-1}E^t\\ -I/2 & E/2\end{pmatrix}. \tag{2.6.17}$$

As expected, g is a representation, $g(W_1W_2)=g(W_1)g(W_2)$. Using the properties of E_M one finds that g is not only an $O(d,d,\mathbf{R})$ rotation, but an element of its discrete subgroup $O(d,d,\mathbf{Z})$ as well. This completes the proof of statement 1.

Statement 3 is easily proved finding the product $\prod g(A_L,A_R,E_M)$ for each generator of $O(d,d,\mathbf{Z})$. We first present the elements in the $d=2$ case, and then we generalize to any d. For $d=2$,

$$\begin{aligned} g_{D_1}=g(r,1,I),\qquad g_{R_1} &= g(r,r,I),\qquad g_{P_{12}}=g(p,p,I),\\ g_{\Theta_{12}} &= g[p,p,E(SU_3)]g(p,p,I),\\ g_{T_{12}} &= g_{R_1}g_{P_{12}}g_D(g_{\Theta_{12}})^{-1}g[p,1,E(SU_3)]g_{P_{12}}g_{R_1}, \end{aligned} \tag{2.6.18}$$

where

$$p=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad r=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\qquad E(SU_3)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}; \tag{2.6.19}$$

$D=D_1D_2$ is the duality symmetry that takes the background matrix to its inverse.

Modulo the permutation P_{12}, which corresponds to an outer automorphism of $SU(2)\times SU(2)$, the list in eq. (2.6.18) is made of gauge transformations. The $SU(2)\times SU(2)$ outer automorphism cannot be viewed, on its own, as a gauge symmetry in a larger group, say $SU(4)$. Neither it is a Weyl symmetry of $SU(4)$ nor of any other larger gauge symmetry (we shall discuss soon what this implies about the identification of the points related by this discrete symmetry).

The automorphisms of the points in moduli space with an enhanced symmetry group $SU(2)\times SU(2)$ and $SU(3)$ (even for specific backgrounds: I and $E(SU_3)$) are sufficient to generate all the elements of $O(d,d,\mathbf{Z})$. This is true if we use the points with such symmetries in any pair of

target space coordinates with indices i, j. In that way one gets D_i, R_i, P_{ij}, T_{ij} and Θ_{ij} using eq. (2.6.18). The group $O(d,d,\mathbf{Z})$ is, therefore, completely generated, and that proves statement 3.

Actually, one can say more. Weyl reflections at the points with an enhanced symmetry $SU(2)^d$ are sufficient in order to get all the gauge transformations. For any simply laced group, G, the following was checked: Let E_M be the background matrix at an enhanced symmetry point G, defined as in eq. (2.4.53) (namely, E_M is the upper-triangular matrix given by the sum of one-half the Cartan matrix plus its antisymmetrization). The $O(d,d,\mathbf{Z})$ transformation that corresponds to the right-handed Weyl reflection of the first simple root ($W_L = 1, W_R \neq 1$) is $R_1 D_1$. The transformation which corresponds to the left-handed Weyl reflection ($W_L \neq 1, W_R = 1$) at the point E_M^t is D_1. These are exactly the $O(d,d,\mathbf{Z})$ transformations that correspond to reflections in the $SU(2)$ case. By a conjugation in $O(d,d,\mathbf{Z})$ one gets all the gauge transformations which correspond to a G-enhanced symmetry point. We conclude, therefore, that the duality relations obtained from different enhanced affine symmetry groups are the same.

The proof of 4a is now very simple. D_1 and $R_1 D_1$ are trivially generated by an even number of generators Θ, P and T. A conjugation in $O(d,d,\mathbf{Z})$ and multiplication preserve the parity of Θ, P, T generators. Therefore, every gauge transformation is even in Θ, P, T. Also, any element which contains even number of Θ, P, T's is in G_{Weyl}. The argument in the previous paragraph also proves statement 2.

We conclude that the group $O(d,d,\mathbf{Z})$ is generated by the groups G_{Weyl} and $\mathbf{Z}_2$. The $\mathbf{Z}_2$ stands for the outer automorphism that permutes two coordinates at the point with an identity background matrix. Therefore, up to a change of names, all the elements of the $O(d,d,\mathbf{Z})$ discrete symmetries are linked to gauge transformations of the kind that was described.

The fact that $O(d,d,\mathbf{Z})$ transformations which are odd in Θ, P, T's are not gauge transformations might be a little surprising. However, there is a simple way to interpret also the odd elements as gauge symmetries in string theory. This is the content of statement 4b, which we are now going to prove.

In string theory there are $26-d$ ($10-d$) space-time coordinates in addition to the d-dimensional internal space. Let g_o (g_e) denote an element which is odd (even) in Θ, P, T's. For $g_o \in O(d,d,\mathbf{Z})$, $g_e = g_o Y_{23,24}$ is an element in $O(d,d,\mathbf{Z}) \times O(2,2,\mathbf{Z})$. $Y_{23,24}$ is any element in $O(2,2,\mathbf{Z})$ which is odd in $\Theta_{23,24}, P_{23,24}$ and $T_{23,24}$. The 23-24 plane is spanned by two space coordinates. The action of g_e on the submoduli, for which the 23'rd and 24'th coordinates are constrained to have an infinite radius, is reduced effectively to g_o. Thus, one may consider g_o as a broken gauge transformation, in the limit where two coordinates are decompactified. Note however, that for $d \geq 21$, the consequence of the identification through a gauge transformation involves the introduction of local symmetries mixing internal and 4-dimensional space-time coordinates.

We have completed the discussion of $O(d,d,\mathbf{Z})$ as a spontaneously broken gauge symmetry in string theory. However, we are left with the $\mathbf{Z}_2$ class of $\mathcal{G}_d$ for which we have not found a general gauge interpretation. These are the symmetries corresponding to worldsheet parity ($B \to -B$ on the background). To discuss properties of this symmetry consider the following.

Recall that a convenient isometric embedding of the moduli space $O(d,d,\mathbf{R})/O(d,\mathbf{R})^2$ in $O(d,d,\mathbf{R})$ is the set of phase-space matrices (see section 2.4, eqs. (2.4.17)-(2.4.23))

$$M = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{2.6.20}$$

Recall also, that the spectrum of the theory is given by $Z^t M Z/2$, where Z is a $2d$-dimensional column vector with integer components. The inner automorphisms (transformations in $O(d,d,\mathbf{R})$) that preserve the spectrum generate the group $O(d,d,\mathbf{Z})$. The transformation that takes B to $-B$ is the outer automorphism

$$M \to \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} M \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.6.21}$$

While the symmetry group $O(d,d,\mathbf{Z})$ corresponds to linear transformations on the currents J (and/or the currents $\bar{J}$), the parity symmetry corresponds to the change $J \leftrightarrow \bar{J}$. Therefore, unlike $O(d,d,\mathbf{Z})$ it cannot be interpreted as a spontaneously broken gauge symmetry.

Finally, let us say few words about the heterotic string. For the heterotic string things are a bit more involved; only elements in the subgroup of $O(d+16,d,\mathbf{Z})$ generated by $g(W_L, W_R = 1, E_M)$ are gauge transformations. (On the submoduli of a zero background gauge field, $A = 0$, they will generate a subgroup made of elements with even number of generators R, Θ, P and T. The reason is that in the $SU(2)$ case only D is a gauge transformation; R and RD act on right-handed currents and as a result the GSO projection in the supersymmetric sector is changed. Now, conjugating I and D we get elements which are even in R, Θ, P, T). However, when W_R is non-trivial, the corresponding symmetry is not linked to a gauge transformation. Yet, we conjecture that all the elements of $O(d+16,d,\mathbf{Z})$ can still be interpreted as spontaneously broken gauge symmetries, in analogy with the bosonic string case.

2.7 Effective Field Theory and $O(6,6,\mathbf{Z})$ Duality

Having seen the rich structure of duality symmetry one would attempt to find an explicit target space realization of it. Here and in section 2.8 we pursue this attempt by studying heterotic $N = 4$ supersymmetric compactifications to $D = 4$. The rational to study local $N = 4$ supersymmetry is that this is the most constrained heterotic background in four dimensions. We adopt the following strategy:

a) We review the construction of $N = 4$ supergravity coupled to matter. The only freedom in this construction is the choice of the gauge structure constants.

b) We find the structure constants of the spin-1 fields relevant to low energy physics by computing their off-shell string 3-point functions.

c) We add to those spin-1 fields other (harmless) degrees of freedom in order to close a completely duality-invariant gauge algebra.

$N = 4$ is the largest supersymmetry that can be possessed by the heterotic string compactified to four dimensions. It is realized when the compact space is a torus. The most general duality group, generated by a gauge symmetry, for torus compactifications of the heterotic string to 4-D space-time, as shown in section 2.5, is $O(6, 22, \mathbf{Z})$. This is the group of $\Gamma^{(6,22)}$ automorphisms which preserve its Lorentzian metric. In particular, when the Lorentzian lattice takes the special form

$$\Gamma^{(6,22)} = \Gamma^{(6,6)} \oplus \Gamma^{(0,16)}, \tag{2.7.1}$$

the Narain compactification reduces to a standard toroidal one, where the background gauge fields in the compactified space are all zero. An obvious subgroup of the duality group is, in this case, $O(6, 6, \mathbf{Z})$, namely, the group of length-preserving automorphisms of $\Gamma^{(6,6)}$.

The group $O(6, 6, \mathbf{Z})$ enjoys an interesting property: it transforms modes that are massless for any $\Gamma^{(6,6)}$ background into themselves [11]. These modes are called "marginal" since they persist and thus correspond to truly marginal deformations of the CFT. If we are interested in low energy physics ($E \ll \alpha'^{-1/2}$), the only relevant interactions are those of the massless modes. All these interactions can be derived from a Low-Energy Effective Action (LEEA), which describes all physics well below the string scale. Since $O(6, 6, \mathbf{Z})$ is a symmetry that (generically) does not mix marginal low energy degrees of freedom with ultramassive ones, the LEEA of a Narain compactification, containing those low-energy degrees of freedom, must be invariant under $O(6, 6, \mathbf{Z})$. When the T^6 torus of the compactification with lattice (2.7.1) is modded out by an appropriate discrete symmetry group one finds the so-called orbifold compactification [86]. In this case, the $N = 4$ supersymmetry of the torus compactification is broken down to $N = 2$, $N = 1$ or $N = 0$. Likewise $O(6, 6, \mathbf{Z})$ breaks down to some non-compact subgroup, e.g. $SL(2, \mathbf{Z})$. This latter group is still an invariance of the CFT of the orbifold compactification, and therefore of the resulting LEEA.

In this section we shall examine the $N = 4$ LEEA, and show that, for compactifications described by lattice (2.7.1), the LEEA is indeed $O(6, 6, \mathbf{Z})$ invariant.

Since the LEEA describes low energy physics, we should include in it only terms with at most two derivatives of the low energy fields. Indeed, higher derivative terms are multiplied by powers of $(\alpha'^{1/2})$, and thus negligible at $E \ll \alpha'^{-1/2}$. Furthermore, the four-dimensional LEEA is $N = 4$ supersymmetric, since it comes from the dimensional reduction of an $N = 1$ theory in ten dimensions [54, 27, 55]. The gauge group, on a generic (non self-dual) background, is

[11] Modes that become massless only on special backgrounds (self-dual radii) do mix with massive ones.

either $E_8 \times E_8 \times U(1)^6$ or $Spin(32)/\mathbf{Z}_2 \times U(1)^6$.[12] The ten-dimensional gauge group $E_8 \times E_8$ or $Spin(32)/\mathbf{Z}_2$ is untouched by $O(6,6,\mathbf{Z})$. In four dimensions and at special values of the background (corresponding to self-dual points w.r.t some element of $O(6,6,\mathbf{Z})$) the $U(1)^6$ factor in the gauge group can be promoted to a non-Abelian group. For instance, at the self-dual point of a circle compactification, $U(1)$ is promoted to $SU(2)$ as was discussed in section 2.6. Since the $SU(2)$ states have nonzero winding number on T^6, they cannot be described by a field-theoretical Kaluza-Klein mechanism. Generically, though, the LEEA containing modes with zero winding or T^6 momenta will offer an adequate description of low energy physics.

The key observation is that an $N = 4$ supersymmetric Lagrangian with at most two derivatives in the fields is completely determined by its gauge group [74, 75, 28, 279]. In other words, the Ward identities of $N = 4$ are so strong as to completely fix the Lagrangian in terms of the gauge-group structure constants.

The most general $N = 4$ supersymmetric Lagrangian coupled to gravity contains two types of multiplets [113]: the gauge matter multiplet and the gravitational multiplet. The gauge matter multiplet is

$$(A_\mu^A, \psi_i^A, Z_{ij}^A), \quad Z_{ij}^A = -Z_{ji}^A, \quad Z^{A\,ij} \equiv (Z_{ij}^A)^* = \frac{1}{2}\epsilon^{ijkl} Z_{kl}^A. \tag{2.7.2}$$

The multiplet contains a vector, two Majorana spinors and six real scalars, all belonging to the adjoint of the gauge group G. The indices $i, j, ... = 1, 2, 3, 4$ are the so-called extension indices: they label the four supersymmetry charges, and span the fundamental representation of $SU(4)$ [113, 16].

The gravitational multiplet contains one spin-2 field (the graviton), four spin-3/2 (gravitini), six spin-1 (graviphotons), four spin-1/2 and one complex scalar

$$(g_{\mu\nu}, \psi_{i\,\mu}, A_{ij\,\mu}, \psi_i, \phi), \quad A_{ij\,\mu} = -A_{ji\,\mu}, \quad A_\mu^{ij} \equiv (A_{ij\,\mu})^* = \frac{1}{2}\epsilon^{ijkl} A_{kl\,\mu}. \tag{2.7.3}$$

The most general Lagrangian containing these multiplets can be found using superconformal techniques [74, 75, 279]. Without entering into details, let us recall some useful notations. The physical (Poincaré) multiplets (2.7.2, 2.7.3) can be re-expressed in terms of the following superconformal multiplets:

a) Superconformal matter

$$(A_\mu^S, \psi_i^S, Z_{ij}^S), \quad S = 1, ..., n + 6. \tag{2.7.4}$$

b) Conformal-supergravity multiplet

$$(g_{\mu\nu}, \psi_{i\,\mu}, \psi_i, \phi). \tag{2.7.5}$$

[12] The extra $U(1)^6$, coming from the heterotic right-movers, corresponds to the graviphotons in the $N = 4$ gravitational supermultiplet.

Here n is the dimension of the gauge group. Multiplets (2.7.4, 2.7.5) contain more degrees of freedom than (2.7.2, 2.7.3). In order to eliminate these degrees of freedom one introduces the constraints [279]

$$\eta_{ST} Z^S_{ij} Z^{T\,kl} = -\delta^{[k}_i \delta^{l]}_j, \quad \eta_{ST} Z^S_{ij} \psi^T_k = 0. \tag{2.7.6}$$

Here $\eta_{ST} = diag(-I_6, +I_{22})$ is the Lorentz metric of signature $(6, 22)$. The physical scalar and spin-1/2 fields have index $S = 7, ..., n+6$, the first six scalars and spin-1/2 are non-propagating auxiliary fields [74, 75, 28, 279].

The most general $N = 4$ Lagrangian can be written in terms of these superconformal fields [74, 75]. We write down here the bosonic Lagrangian only. Fermionic terms, obviously, are uniquely fixed by supersymmetry[13].

$$\begin{aligned} e^{-1}L &= \eta_{RS}\left\{-\frac{1}{4}\frac{\psi}{\phi}F^{+R}_{\mu\nu}F^{\mu\nu\,+S} - \frac{1}{4}D_\mu Z^R_{ij} D^\mu Z^{ij\,S}\right. \\ &\quad \left. +\frac{1}{2}\frac{\phi^*}{\phi}K^{+R}_{\mu\nu}K^{\mu\nu\,+S}\right\} - \frac{1}{4}R(\omega) + 4\Re D_\mu\phi D^\mu\psi^* \\ &\quad +\frac{1}{8}E_{ij}E^{ij} + \eta_{RS}\left\{\frac{1}{4}|\phi|^2 W^{j\,R}_i W^{i\,S}_j + \frac{1}{3}E^{ij}X^{RS}_{ij}\phi^*\right\} \end{aligned} \tag{2.7.7}$$

In eq. (2.7.7) $\Re\phi\psi^* = 1$, $e \equiv \det e^a_\mu$, and we have used the following notation [74, 75]

$$\begin{aligned} F^{+\,R}_{\mu\nu} &= \frac{1}{2}F^R_{\mu\nu} + \frac{1}{4}e^{-1}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma\,R}, \quad D_\mu Z^P_{ij} = \partial_\mu Z^P_{ij} + f^P{}_{QR}A^Q_\mu Z^R_{ij} \\ X^{RS}_{ij} &= Z^T_{ik}Z^{kl\,R}Z^U_{lj}f^S{}_{TU}, \quad W^{j\,R}_i = Z^S_{ik}Z^{kj\,T}f^R{}_{ST} \\ E_{ij} &= -\frac{4}{3}\phi^*\eta_{RS}X^{RS}_{ij}, \quad K^{+\,R}_{\mu\nu} = \frac{1}{\phi^*}Z^{ij\,R}F^{+\,S}_{\mu\nu}Z^T_{ij}\eta_{ST}. \end{aligned} \tag{2.7.8}$$

As usual, $f^P{}_{QR}$ denote the gauge-group structure constants. The fields Z^S_{ij} include 36 non-propagating scalars. Notice that the indices ij parametrize the antisymmetric representation of $SU(4)$ (which is isomorphic to the vectorial of $SO(6)$). The $n+6$ scalars Z^S_{ij}, due to the constraints (2.7.6) parametrize the coset manifold $O(6,n)/(O(6)\times O(n))$. The indices S, T, etc. are both in the adjoint of the gauge group, and in the fundamental of $O(6,n)$. Therefore the allowed gauge groups G of $N = 4$ supergravity are those satisfying $Adj\,G \subset Vect\,SO(6,n)$. Notice that in eq. (2.7.7) the n matter gauge fields and the six graviphotons group together naturally in $F^S_{\mu\nu}$, $S = 1, ..., n+6$. This grouping is due to the fact that the "parent" conformal-supersymmetry spin-1 fields all belong to superconformal matter, and not to the multiplet of superconformal gravity (2.7.5).

It must be noticed that the gauge group can be non-compact if $Adj\,G$ does *not* belong to $SO(6)\times SO(n)$, that is, if the gauge group acts nontrivially on both physical *and* compensating

[13] This Lagrangian could be further generalized [74, 75] at the price of reducing the kinetic-term duality group [121]. This goes exactly in the opposite direction of our program of finding the *maximally* duality-invariant LEEA.

multiplets (2.7.4). A generic toroidal compactification on $\Gamma^{(6,6)} \oplus \Gamma^{(0,16)}$, on the other hand, yields the compact gauge group $\mathcal{G} = U(1)^6 \times G$, with G either $E_8 \times E_8$ or $Spin(32)/\mathbf{Z}_2$. In this case

$$f^P{}_{QR} = 0, \quad P, Q, R = 1, ..., 12. \tag{2.7.9}$$

Lagrangian (2.7.7), when all $f^P{}_{QR} = 0$, is invariant under the following non-compact transformation

$$Z^S_{ij} \to \Lambda^S_T Z^T_{ij}, \quad F^S_{\mu\nu} \to \Lambda^S_T F^T_{\mu\nu}, \quad \Lambda^P_S \Lambda^Q_T \eta_{PQ} = \eta_{ST}. \tag{2.7.10}$$

This $O(6,n)$ symmetry is explicitly broken by the structure-constant dependent terms. When $\mathcal{G} = U(1)^6 \times G$ the residual non-compact symmetry is $O(6,6)$.

This result means that the LEEA involving massless modes possesses indeed the $O(6,6,\mathbf{Z})$ symmetry we found in the worldsheet analysis of sections 2.4, 2.5. In fact, the residual symmetry of the LEEA eq. (2.7.7) is even larger than expected: it is a continuous group $O(6,6,\mathbf{R})$ instead of a discrete one[14]. When higher derivative terms ($\mathcal{O}(\alpha')$) are included in the LEEA, the invariance group is expected to break down to $O(6,6,\mathbf{Z})$. This latter group, being an invariance of the underlying worldsheet conformal theory, is an exact symmetry of the LEEA, to all orders in α'.

We have just found how easy it is to write down a LEEA invariant under the $O(6,6,\mathbf{Z})$ subgroup of the full duality group $O(6,22,\mathbf{Z})$. This happens because $O(6,6,\mathbf{Z})$ transforms the marginal modes into themselves. On the other hand, constructing a LEEA invariant under $O(6,22,\mathbf{Z})$ is a more complex endeavor. Indeed, within $O(6,22,\mathbf{Z})$ there exist elements transforming states that are massless for any background of the form $\Gamma^{(6,6)} \oplus \Gamma^{(0,16)}$ into ultra-massive ones. This fact is due to the non-Abelian character of the gauge group. Explicitly, $E_8 \times E_8$ states outside the Cartan subalgebra are mapped into massive modes under generic $O(6,22,\mathbf{Z})$ transformations. Moreover, the typical orbit of a string state under $O(6,22,\mathbf{Z})$ contains an infinite number of points. Still, since string backgrounds related by duality yield the same physics, we do expect an $O(6,22,\mathbf{Z})$-invariant LEEA to exist. This LEEA may contain an infinite number of fields, and even ghosts, provided their masses are always of the order of the compactification scale R, or larger. The essential requirement we must impose on the $O(6,22,\mathbf{Z})$-invariant LEEA is to reproduce the correct physics at energies $E \ll R^{-1}, \alpha'^{-1}R$. The next section is devoted to the construction of such LEEA.

[14] The dilaton $\Re S \equiv \Re(\psi/\phi) = |\phi|^{-2}$ in (2.7.7) is the duality invariant one (see subsection 4.2.4 for more details).

2.8 A Completely Duality-Invariant Low-Energy Effective Action for the $N = 4$ Heterotic String

In the moduli space of toroidal compactifications there is an infinite number of special points where massive modes become massless. These points correspond to systems with an enhanced affine symmetry. Massless fields become massive at the neighborhood of such points by the Higgs mechanism. The generalized duality group mixes massless modes with massive modes that might become massless at some point of the moduli space. Therefore, the LEEA should contain *all* the fields which correspond to string states that become massless at some point of the moduli space. The number of such fields is infinite, and on a given background all except a finite number of them are very massive. In this section we present the relevant structure constants of the infinite dimensional "Duality Invariant String Gauge algebra" (DISG). We justify the name given to this group by showing that the Higgs mechanism in the LEEA coincides with the stringy Higgs mechanism. Namely, the masses of the LEEA scalar fields coincide with the string spectrum at the particular background. We also show the invariance of the LEEA under the action of the generalized duality group $O(6,22,\mathbf{Z})$, and the structure of the DISG algebra. Our analysis follows closely ref. [144]

Since the form of a gauged $D = 4$, $N = 4$ supergravity (with at most two space-time derivatives) coupled to matter multiplets is determined by the knowledge of the gauge group [74, 75, 279, 28], one has only the freedom to choose the right number of matter multiplets and the correct gauge group in order to figure out completely the LEEA of the string. Let us start from the solution and present the matter multiplets and the structure constants of the DISG algebra which are relevant for $D = 4$ low energy physics.

The scalar fields which are relevant for $D = 4$ low energy physics Z_a^S, $a = 1, ..., 6$ [15], $S = 1, ..., \infty$ will be labeled as follows:

$$\begin{gathered} Z_a^b,\ Z_a^I,\ Z_a^p,\ Z_a^{-p} \equiv \bar{Z}_a^p, \\ p \in \Gamma^{(6,22)},\quad p\eta p = 2,\quad a,b = 1,..,6,\quad I = 7,..,28, \end{gathered} \tag{2.8.1}$$

where $\Gamma^{(6,22)}$ is an even self-dual Lorentzian lattice of signature $(6,22)$. The "momentum" $p = (p_L, p_R)$ has 6 left-handed components p_L^a, $a = 1, ..., 6$, and 22 right-handed components p_R^I, $I = 7, ..., 28$. This is the Narain lattice of the string toroidal compactification [236, 238]. The scalar product is Lorentzian, $p\eta q = -p_L^a q_L^a + p_R^I q_R^I$, where $L(R)$ denote left-handed (right-handed) momenta. The indices (b, I) refer to the Cartan Sub-Algebra (CSA), while the p indices are of the Lorentzian length two generalized roots (and therefore their number is infinite). These fields live on the infinite dimensional coset space

$$\frac{O(6,\infty,\mathbf{R})}{O(6,\mathbf{R}) \times O(\infty,\mathbf{R})}. \tag{2.8.2}$$

[15] We used here the isomorphism between the vector representation of $SO(6)$, indexed by a, and the antisymmetric of $SU(4)$, indexed by ij.

The structure constants of the gauge group which are relevant for $D = 4$ low energy physics are

$$f^r_{pq} = \varepsilon(p,q)\delta^r_{p+q}, \quad f^I_{p,-p} = -ip^I, \quad f^q_{Ip} = -ip_I\delta^q_p, \quad f^b_{p,-p} = -ip^b,$$
$$f^q_{bp} = -ip_b\delta^q_p, \quad p\eta p = q\eta q = r\eta r = 2. \tag{2.8.3}$$

The two cocycle $\varepsilon(p,q)$ satisfies the identities

$$\begin{aligned} \varepsilon(p,q)\varepsilon(p+q,r) &= \varepsilon(p,q+r)\varepsilon(q,r), \\ \varepsilon(p,q) &= (-1)^{p\eta q}\varepsilon(q,p), \quad \varepsilon(p,-p) = 1, \end{aligned} \tag{2.8.4}$$

and

$$\varepsilon(p,q) = \varepsilon(q,-p-q). \tag{2.8.5}$$

When p and q are the roots of a finite dimensional gauge group g, the $\varepsilon(p,q)$ reduce to the structure constants of g in the Cartan-Weyl basis [165, 19, 117]. As demonstrated in [117] $\varepsilon(p,q)$ can be extended to all vectors in $\Gamma^{(6,22)}$. An elegant explicit construction of $\varepsilon(p,q)$ has been given in [155].

The set of structure constants in (2.8.3) is not sufficient to close a Lie algebra. The Lie algebra that contains all fields presented in (2.8.3) and preserves the $D = 4$ low energy physics is the DISG algebra. We will elaborate on the DISG algebra later on. Meanwhile, we will consider only the sub-set of generators discussed above.

The metric η is easily extended from the CSA to the generators which are relevant for the $D = 4$ low energy theory by defining $\eta_{p,-p} = \eta_{-p,p} = 1$, and setting all other components to zero. By lowering with this metric all indices, one finds that the structure constants f_{QST} are completely antisymmetric, due to (2.8.5).

Using the infinite dimensional DISG as the gauge group of the low energy $D = 4$, $N = 4$ supergravity theory means to use the structure constants (2.8.3) in the action (2.7.7). The covariant derivatives and the scalar potential are obtained upon substituting the structure constants (2.8.3) into equations (2.7.7) and (2.7.8). Let us show, at first, that the Higgs mechanism of the LEEA coincides with the stringy Higgs mechanism.

The simplest zero cosmological constant minima of the action (2.7.7) lie in the CSA. The value of the cosmological constant is found by writing down the scalar potential

$$V = \frac{1}{4}|\phi|^2 Z^{QU}Z^{SV}(\eta^{TW} + \frac{2}{3}Z^{TW})f_{QST}f_{UVW}, \quad Z^{PQ} \equiv Z^P_a Z^Q_a. \tag{2.8.6}$$

Let us expand the scalar fields around a VEV in the CSA,

$$Z^S_a = C^b_a\delta^S_b + C^I_a\delta^S_I + \zeta^S_a, \tag{2.8.7}$$

where C_a^b, C_a^I, $a,b = 1,..,6$, $I = 7,..,28$, are constants and the constraint (2.7.6) is satisfied. A rotation of Z_a^S to

$$(Z')_a^{S'} = Z_a^S M^{S'}_{\ S}, \tag{2.8.8}$$

where $M \in O(6,\infty,\mathbf{R})$, preserves the scalar quadratic constraint (2.7.6) (which now reads $\eta_{ST} Z_a^S Z_b^T = -\delta_{ab}$). Changing the VEV of the scalar fields to new ones in the CSA can be done by a rotation in $O(6,22,\mathbf{R}) \subset O(6,\infty,\mathbf{R})$.

Under the orthogonal transformation (2.8.8), the scalar potential V transforms into

$$\begin{aligned} V' &= \frac{1}{4}|\phi|^2 (Z')^{QU}(Z')^{SV}(\eta^{TW} + \frac{2}{3}(Z')^{TW}) f_{QST} f_{UVW} \\ &= \frac{1}{4} Z^{QU} Z^{SV} (\eta^{TW} + \frac{2}{3} Z^{TW}) f'_{QST} f'_{UVW}, \end{aligned} \tag{2.8.9}$$

where

$$f'_{QST} = (M^{-1})_T^{\ T'} (M^{-1})_Q^{\ Q'} f_{Q'S'T'} M^{S'}_{\ S}. \tag{2.8.10}$$

Thus changing the VEV of the Higgs fields in the CSA is equivalent to a transformation of the structure constants (2.8.3), given by an $O(6,22,\mathbf{R})$ rotation of $\Gamma^{(6,22)}$ (i.e. a rotation of the "momenta" labels p). This transformation is an isomorphism of the gauge algebra. Finding the mass spectrum of the theory on an arbitrary VEV is now very simple.

Using the field redefinition described above, one can always choose a zero cosmological constant minimum to lie at

$$\begin{gathered} \phi = 1, \quad Z_a^b = \delta_a^b, \qquad a,b = 1,..,6, \\ Z_a^I = 0, \ I = 7,...,28, \quad Z_a^p = 0 \ \ \forall p \in \Gamma^{(6,22)}. \end{gathered} \tag{2.8.11}$$

Around this point the mass spectrum of the effective action is exactly that predicted by string theory. This is easily checked by looking at the mass term of the gauge boson A_μ^p. Using eqs. (2.7.8, 2.8.3, 2.8.7) one finds

$$\begin{aligned} \frac{1}{2}\eta_{pq} D_\mu Z_a^p D^\mu Z^{aq} &= \frac{1}{2}\partial_\mu \zeta_a^p \partial^\mu \zeta^{a\,-p} + i\partial^\mu \zeta_a^p f^a_{pq} A_\mu^q - \frac{1}{2} A_\mu^q f^p_{qa} A^{\mu r} f^a_{rp} + ... \\ &= \frac{1}{2}\partial_\mu \zeta_a^p \partial^\mu \zeta^{a\,-p} + p^a \partial^\mu \zeta_a^p A_\mu^{-p} + \frac{1}{2} p_a p^a A_\mu^p A^{\mu\,-p} + ... \end{aligned} \tag{2.8.12}$$

In (2.8.12) we chose the string tension to be $\alpha' = 1$, and '...' stands for higher-order terms.

The gauge-vector kinetic term is $F_{\mu\nu}F^{\mu\nu}/4$, thus the mass of the gauge boson A_μ^p is

$$m_p^2 = p_a p^a, \quad a = 1,...,6. \tag{2.8.13}$$

This is exactly the mass of a string vector boson with internal momentum $p = (p_L, p_R)$, s.t. the components of p_L are p_a [16]. The field $p^a\zeta_a^p$ in equation (2.8.12) is the Goldstone boson. The appearance of the correct masses and the Goldstone boson can be rechecked by expanding the scalar potential (2.8.6) up to quadratic terms in the fluctuations around the point (2.8.11). One finds:

$$V = (p^a p_a)\zeta^{bp}\zeta_b^{-p} - p^a p^b \zeta_a^p \zeta_b^{-p} + o(\zeta^3). \tag{2.8.14}$$

Equation (2.8.14) gives five scalars with mass $p^a p_a$ and one massless scalar for each nonzero p. The massless scalars are, naturally, the Goldstone bosons of the broken generators of the gauge group.

It is important to realize that, since the infinite dimensional gauge group acts nontrivially on the compensators Z_b^a, and since the latter can never vanish, because of constraint (2.7.6), the gauge symmetry is always broken to some finite dimensional rank 22 group g times the $U(1)^6$ (originated by the left-handed sector of the string). The unbroken gauge group is generically $U(1)^{22}$, however, it is extended to bigger rank 22 groups at special VEVs [17]. In the neighborhood of such VEV the $N = 4$ supergravity is reduced to a gauged $D = 4$, $N = 4$ supergravity with a finite dimensional gauge group g.

The condition for a transformation M in (2.8.8) to be a residual symmetry of the LEEA is

$$f'_{QST} = f_{QST}, \tag{2.8.15}$$

where f'_{QST} is defined in (2.8.10), i.e. M is an automorphism of the gauge group. By construction, the full duality group $O(6, 22, \mathbf{Z})$ is a symmetry of the LEEA. This is because $O(6, 22, \mathbf{Z})$ *is the Weyl group of the gauge algebra* [18].

We pass now to the determination of the gauge group of the duality invariant low-energy effective action. As mentioned before, duality implies that massive states are to be included as well. For example, we must include those states which become massless on some background, together with all states related to them by duality. It should be stressed, on the other hand, that the LEEA, by definition, gives the correct string dynamics only for those processes involving light particles, with energies far below the string (or Planck) scale. For this reason the complete spectrum of the LEEA needs not (and indeed does not) coincide with the string spectrum. Only states related by duality to massless ones must coincide.

[16] The string vertex corresponding to the field A_μ^p is $\psi_\mu e^{i(p_{La}x_L^a + p_{RI}x_R^I)}$. The mass and left-right level matching condition are given by: $m^2 = m_L^2 = m_R^2$, where $\frac{1}{2}m_L^2 = \frac{1}{2} + \frac{1}{2}p_L^2 - \frac{1}{2} = \frac{1}{2}p_L^2$ (the first 1/2 is the conformal dimension of the worldsheet fermion and the $-1/2$ is the normal ordering constant in the NS sector) and $\frac{1}{2}m_R^2 = \frac{1}{2}p_R^2 - 1$ (the -1 is the normal ordering constant of the bosonic right-handed sector).

[17] In string theory this corresponds to points in the moduli space with an enhanced affine symmetry.

[18] For gauged $N = 4$ supergravity with a finite gauge group g the automorphism group of g is a subgroup of $O(6, 22, \mathbf{Z})$ which leaves the enhanced symmetry point in the moduli space fixed [141], as discussed from the worldsheet point of view in section 2.6.

The strategy for determining the LEEA reads as follows. We determine first the string-theoretical three-point functions of those spin-one fields A^S_μ that become massless at some point of the moduli space. We then take the low-momentum limit of these three-point functions by defining an appropriate off-shell continuation. From these three-point functions we extract a set of structure constants f_{STU}. Finally, we find a Lie algebra whose structure constants coincide with f_{STU} when evaluated on the original states A^S_μ.

The spin-one states which become massless at some point of the moduli space are of two kinds. The first one is made of vectors which are massless on all backgrounds

$$\begin{aligned}\varepsilon_\mu V^{a\mu}(\mathbf{p}) &= \int d\theta d^2z \varepsilon_\mu D_\theta X^a \bar\partial X^\mu \exp(ip_\nu X^\nu)(\theta,z,\bar z), \quad a=1,..,6,\\ \varepsilon_\mu V^{I\mu}(\mathbf{p}) &= \int d\theta d^2z \varepsilon_\mu D_\theta X^\mu \bar\partial X^I \exp(ip_\nu X^\nu)(\theta,z,\bar z), \quad I=7,..,28.\end{aligned} \tag{2.8.16}$$

Here ε_μ is a polarization vector obeying $\varepsilon_\mu p^\mu = 0$; the vector $\mathbf{p}$ is the space-time momentum with components p_μ; X is a heterotic superfield, i.e. its left-handed part is 2-d supersymmetric, $X(z,\bar z,\theta) = X_L(\bar z) + X_R(z) + \theta\psi(z)$, and D_θ is a spinorial derivative, $D_\theta = \partial_\theta + \theta\partial$.

The second one includes all spin-one fields which, at particular points of the moduli space, extend the Abelian $U(1)^{28}$ symmetry generated by (2.8.16) to larger non-Abelian symmetries. The vertices corresponding to these fields are

$$\begin{aligned}\varepsilon_\mu V^{p\mu}(\mathbf{p}) &= \int d\theta d^2z C(p) \varepsilon_\mu D_\theta X^\mu \exp(ip_a X^a_L + ip_I X^I_R + ip_\nu X^\nu)(\theta,z,\bar z),\\ &\quad -p_a^2 + p_I^2 = 2.\end{aligned} \tag{2.8.17}$$

In eq. (2.8.17) $C(p)$ is an operator defined for every internal lattice vector p and obeying [117, 162]

$$C(p)C(q) = \varepsilon(p,q)C(p+q). \tag{2.8.18}$$

The two cocycle $\varepsilon(p,q)$ satisfies the compatibility conditions (2.8.4) and (2.8.5). Its presence is necessary in order to reproduce the correct commutators for the unbroken Lie algebra on the vectors $p_a = 0$, $p_I^2 = 2$ [117].

The spin one states (2.8.16) and (2.8.17) give rise to (broken) gauge symmetries, at least in that they give rise to the correct (on-shell) Ward identities [158, 277, 220, 218]

$$p_\mu V^{a\mu} \approx 0, \quad p_\mu V^{I\mu} \approx 0, \quad p_\mu V^{p\mu} + p_a V^{pa} \approx 0. \tag{2.8.19}$$

Here $\approx$ denotes an on-shell equality, and V^{pa} is the scalar-field vertex

$$\begin{aligned}V^{pa}(\mathbf{p}) &= \int d\theta d^2z C(p) D_\theta X^a \exp(ip_a X^a_L + ip_I X^I_R + ip_\nu X^\nu)(\theta,z,\bar z),\\ &\quad -p_a^2 + p_I^2 = 2.\end{aligned} \tag{2.8.20}$$

From eq. (2.8.19) one can recover an off-shell gauge invariance of the effective action, as shown in [244].

Besides (2.8.16) and (2.8.17), other fields may become (almost) massless in the appropriate regions of the moduli space. They are characterized by lattice vectors of zero (Lorentzian) norm $-p_a^2 + p_I^2 = 0$. These states become exactly massless at the boundary of the moduli space, that is (for example) when some of the compactification radii $R \to \infty$. Since this limit gives rise to a higher dimensional theory, we shall not include those fields. Our aim is only to build a four-dimensional effective action. This action makes sense only when the dimensions of the compactified space are $O(M_{Pl}^{-1})$, where M_{Pl} is the Planck mass.

Let us evaluate the relevant three-point functions involving the vector fields (2.8.16) and (2.8.17). We define the off-shell three-point function according to ref. [211, 210] as

$$\begin{aligned} V_{ijk} &= \lim_{\epsilon\to 0}\langle 0|h^2 \circ V_i(\epsilon) h \circ V_j(\epsilon) V_k(\epsilon)|0\rangle \\ &= \lim_{\epsilon\to 0}[h'(h(\epsilon))h'(\epsilon)]^{d_i} h'(\epsilon)^{d_j} \langle 0|V_i(h^2(\epsilon))V_j(h(\epsilon))V_k(\epsilon)|0\rangle. \end{aligned} \tag{2.8.21}$$

In eq. (2.8.21) h denotes the $SL(2,\mathbf{C})$ transformation sending $\infty \to 0$, $0 \to 1$, $1 \to \infty$, i.e.

$$h(z,\bar z) = (h,\bar h) = \left(\frac{1}{1-z}, \frac{1}{1-\bar z}\right), \tag{2.8.22}$$

and $h' = |\partial h|^2$. The suffix d_i is the conformal weight of the vertex V_i, which also includes reparametrization (and super-reparametrization) ghosts. Obviously $d_i = 0$ for on-shell vertices.

In eqs. (2.8.16) and (2.8.17) we represented all vector vertices in the zero-ghost picture [119]. Actually it is convenient to use several different pictures simultaneously. In the -1 picture, the vertices in eq. (2.8.17) read, e.g.

$$\varepsilon_\mu V^{\mu p}_{(-1)}(z,\bar z) = \varepsilon_\mu C(p)\psi^\mu \exp(-\phi)\exp(ip_a X_L^a + ip_I X_R^I + ip_\nu X^\nu) c\bar c(z,\bar z). \tag{2.8.23}$$

Here c and $\bar c$ are the reparametrization ghosts, and $\phi(z,\bar z)$ is the bosonized supersymmetry ghost [119]. $\psi^\mu(z)$ is the fermionic superpartner of $\partial X^\mu(z)$.

Let us evaluate

$$V_{PQR} = \lim_{\epsilon\to 0}\langle 0|h^2 \circ \varepsilon V^p_{(0)}(\mathbf{p})(\epsilon) h \circ \varepsilon V^q_{(-1)}(\mathbf{q})(\epsilon)\varepsilon V^r_{(-1)}(\mathbf{r})(\epsilon)|0\rangle. \tag{2.8.24}$$

We need the following correlation functions

$$\begin{aligned} \langle \exp(-\phi)(\frac{1}{1-\epsilon})\exp(-\phi)(\epsilon)\rangle &= 1 + O(\epsilon), \\ \langle (p\cdot\psi)\psi^\mu(1/\epsilon)\psi^\rho(1)\psi^\sigma(0)\rangle &= \epsilon^2(g^{\mu\rho}p^\sigma - g^{\mu\sigma}p^\rho) \\ &\quad + O(\epsilon^3), \\ \langle \exp(iPX)(1/\epsilon)\exp(iQX)(1)\exp(iRX)(0)\rangle &= \epsilon^{P^2}[1+O(\epsilon)], \\ \langle \partial X^\mu \exp(iPX)(1/\epsilon)\psi^\rho \exp(iQX)(1)\psi^\sigma \exp(iRX)(0)\rangle &= i[-p^\mu \epsilon^{P^2+1} + \\ q^\mu \epsilon^{P^2+2}]g^{\rho\sigma} + O(\epsilon^{P^2+3}). \end{aligned} \tag{2.8.25}$$

Here $P+Q+R=0$, where $P=(p_a,p_I,p_\nu)$ is the total momentum vector, and $g^{\mu\nu}$ is the space-time metric. The $\langle\rangle$ in the first line of (2.8.25) includes the insertion of the charge at infinity $\exp 2\phi_0$, where ϕ_0 is the zero mode of ϕ. Substituting eqs. (2.8.25) in definition (2.8.24) we find, taking into account the transversality of the polarization vectors ε,

$$V_{PQR}=i\varepsilon_{\mu p}\varepsilon_{\rho q}\varepsilon_{\sigma r}(g^{\mu\rho}p^\sigma+g^{\mu\sigma}r^\rho+g^{\rho\sigma}q^\mu)\varepsilon(p,q). \tag{2.8.26}$$

In (2.8.26) the two cocycle $\varepsilon(p,q)$ is a function of only the internal momenta, as $C(p)$ in eqs. (2.8.17, 2.8.18) depends only on the internal lattice vectors.

The other three-point functions we need are

$$V_{aQR}=\lim_{\epsilon\to 0}\langle 0|h^2\circ V^a_{(0)}(\mathbf{p})(\epsilon)h\circ\varepsilon V^q_{(-1)}(\mathbf{q})(\epsilon)\varepsilon V^r_{(-1)}(\mathbf{r})(\epsilon)|0\rangle, \tag{2.8.27}$$

and

$$V_{IQR}=\lim_{\epsilon\to 0}\langle 0|h^2\circ V^I_{(0)}(\mathbf{p})(\epsilon)h\circ\varepsilon V^q_{(-1)}(\mathbf{q})(\epsilon)\varepsilon V^r_{(-1)}(\mathbf{r})(\epsilon)|0\rangle. \tag{2.8.28}$$

By calculations similar to those leading to eq. (2.8.26), and since

$$\begin{aligned}\langle\bar\partial X^\mu\partial X^a\exp(iPX)(1/\epsilon)\psi^\rho\exp(iQX)(1)\psi^\sigma\exp(iRX)(0)\rangle \;&=\\ -g^{\rho\sigma}q^aq^\mu\epsilon^{P^2+2}+O(\epsilon^{P^2+3}),&\end{aligned} \tag{2.8.29}$$

we find

$$\begin{aligned} V_{aQR} &= -\varepsilon_{\mu p}\varepsilon_{\rho q}\varepsilon_{\sigma r}g^{\rho\sigma}q^\mu q^a,\\ V_{IQR} &= -\varepsilon_{\mu p}\varepsilon_{\rho q}\varepsilon_{\sigma r}(g^{\mu\rho}p^\sigma+g^{\mu\sigma}r^\rho+g^{\rho\sigma}q^\mu)q^I,\\ P+Q+R &= 0,\quad P=(0,\mathbf{p}).\end{aligned} \tag{2.8.30}$$

Now, we must compare eqs. (2.8.26) and (2.8.30) with the three-point functions one gets from an $N=4$ supergravity Lagrangian discussed in section 2.7. By eliminating auxiliary fields and expanding the $N=4$ Lagrangian around the background (2.8.11) (see for example [28]), we find the kinetic term

$$\frac{1}{4}\sum_a F^a_{\mu\nu}F^{a\,\mu\nu}+\frac{1}{4}\sum_{K,K'}F^K_{\mu\nu}F^{K'\,\mu\nu}\eta_{KK'}. \tag{2.8.31}$$

Notice that the graviphotons A^a_μ appear with positive signature. The index $K=(I,p)$ labels all spin-one gauge fields belonging to vector multiplets. The term, trilinear in the vector fields, that we get from (2.8.31) is

$$\begin{aligned}&\partial^\mu A^{\nu a}f^a_{KL}A^K_\mu A^L_\nu+\partial^\mu A^{\nu K}f^{K'}_{aL}A^a_\mu A^L_\nu\eta_{KK'}+\\ &\partial^\mu A^{\nu K}f^{K'}_{La}A^L_\mu A^a_\nu\eta_{KK'}+\partial^\mu A^{\nu K}f^{K'}_{LM}A^L_\mu A^M_\nu\eta_{KK'}.\end{aligned} \tag{2.8.32}$$

Recalling that the indices a, K are lowered with the matrix η_{ST} which obeys $\eta_{ab} = -\delta_{ab}$, $a = 1, .., 6$, and that f_{STU} must be completely antisymmetric [74, 75, 28, 143], we cast (2.8.32) in the following form

$$- \partial^\mu A^{\nu K} f_{aKL} A^a_\mu A^L_\nu + \partial^\mu A^{\nu K} f_{KLM} A^L_\mu A^M_\nu . \tag{2.8.33}$$

By substituting to the A^S_μ of eq. (2.8.33) the appropriate transverse plane waves, we find the three-point functions

$$\begin{aligned} V_{aKL} &= -i\varepsilon_{\mu p}\varepsilon_{\rho q}\varepsilon_{\sigma r} g^{\rho\sigma} q^\mu f_{aKL}, \\ V_{KLM} &= i\varepsilon_{\mu p}\varepsilon_{\rho q}\varepsilon_{\sigma r}(g^{\rho\sigma} q^\mu + g^{\mu\sigma} r^\rho + g^{\mu\rho} p^\sigma) f_{KLM}. \end{aligned} \tag{2.8.34}$$

Comparing eq. (2.8.34) with eqs. (2.8.26) and (2.8.30) we get

$$f_{aqr} = -iq^a, \quad f_{Iqr} = -iq^I, \quad f_{pqr} = \varepsilon(p, q), \tag{2.8.35}$$

which are the structure constants in eq. (2.8.3).

Next, we ought to find a Lie algebra whose structure constants include those given in (2.8.35). This algebra can be defined as follows [155, 143, 144]. Let us consider twenty-eight free bosonic fields with correlation functions

$$\langle X^A(z) X^B(w)\rangle = -\eta^{AB} \log(z - w), \quad A, B = 1, .., 28. \tag{2.8.36}$$

Here $\eta^{AB} = \mathrm{diag}(-1^6, 1^{22})$. Notice that these fields are *chiral*.

Let us further compactify the bosonic fields on a $(6, 22)$ Lorentzian even self-dual lattice $\Gamma^{(6,22)}$. Let us denote, as before, with $X^a(z)$ the six negative-metric bosons, and with $X^I(z)$ the other twenty two positive metric bosons. For each lattice vector of Lorentzian norm $p^A p^B \eta_{AB} = 2$, we associate a *chiral* vertex operator

$$V^p = \oint \frac{dz}{2\pi i} C(p) \exp(ip_A X^A)(z), \quad p^A p^B \eta_{AB} = 2. \tag{2.8.37}$$

The vertices (2.8.37), together with

$$V^A = \oint \frac{dz}{2\pi i} \partial X^A(z), \quad A = 1, .., 28, \tag{2.8.38}$$

give rise to a Lie algebra $\mathbf{g}$, whose elements are (2.8.37) and (2.8.38) together with all possible multiple commutators of the V^p. This algebra is a particular example of (indefinite-signature) lattice algebra [155]. In some cases lattice algebras reduce to hyperbolic, affine or finite Kac-Moody algebras [155, 180]. The vertices V^A span a maximal commuting subalgebra [155], which plays the role of the Cartan subalgebra. The rank of $\mathbf{g}$ is thus finite (and equal to 28, in our example).

Two facts ought to be noticed now. The first one is that, as shown in the following, there exist a **g** invariant non-degenerate two form η, with at least six negative eigenvalues, which reduces to η_{AB} on the Cartan subalgebra. This implies the identity

$$f^V_{ST}\eta_{VU} + f^V_{SU}\eta_{TV} = 0. \tag{2.8.39}$$

Here S, T, U, V label the generators of **g**. Identity (2.8.39) is indispensable for the construction of an $N = 4$ supersymmetric Lagrangian [74, 75, 28, 279].

The second fact we should emphasize is that in **g** there are many states actually not present in the string spectrum. This is not a serious problem for our construction, anyway. We are in fact interested in constructing only a duality-invariant *low-energy* effective action. All new states introduced by the definition of **g** have masses $O(M_{Pl})$ for all values of the lattice momenta p^A corresponding to four-dimensional compactifications of the heterotic string. The only region in which some of the additional states may become light corresponds to the decompactification limit mentioned earlier in this section. In this limit some states with momenta obeying $p^A p^B \eta_{AB} = 0$ may become almost massless. On the other hand, a 4-D low-energy effective action cannot be expected to correctly describe a higher dimensional theory. The Kaluza-Klein spectrum is indeed truncated down to the lightest states from the very beginning. A phenomenon similar to the one encountered here has already been noticed in the past, in connection with the study of string effective actions. In [90, 51] e.g., it was shown that by adding higher derivative terms to the effective Lagrangian of both the bosonic [90] and heterotic [90, 51] strings, additional states of mass $O(M_{Pl})$, as well as ultramassive ghosts, may propagate on a generic background. This is the price paid for truncating the higher spin modes of the string.

We pass now to a more detailed study of the DISG algebra **g**. The simplest new states that one finds by taking the commutator of two vertices (2.8.37) are

$$\begin{aligned} [V^p, V^q] \;\; &= \;\; \varepsilon(p,q) \oint \frac{dz}{2\pi i} C(p+q) p_B \partial X^B \exp i(p_A + q_A) X^A(z), \\ &\qquad (p_A + q_A)(p^A + q^A) = 0 \quad A, B = 1, .., 28. \end{aligned} \tag{2.8.40}$$

In other terms, by taking the commutator of two vertices (2.8.37), corresponding to momenta obeying $p\eta q = -2$, one finds "photon" states of the form

$$\oint \frac{dz}{2\pi i} C(p) \xi_A \partial X^A \exp(i p_A X^A)(z), \quad p_A p^A = 0, \quad p_A \xi^A = 0. \tag{2.8.41}$$

The mass of the states one gets from formula (2.8.13), is only small in the decompactification limit, in which, as previously stressed, the whole effective action approach looses meaning. The low energy dynamics of strings is therefore not affected by the presence of these new "photon" states.

The states with $p\eta p = -2$, whose mass square is always larger than M^2_{Pl}, read

$$V^p(\xi_{AB}, \xi_A) = \oint \frac{dz}{2\pi i} C(p) (\xi_{AB} \partial X^A \partial X^B + \xi_A \partial^2 X^A) \exp(i p_A X^A)(z), \tag{2.8.42}$$

where the coefficients ξ_{AB}, ξ_A obey

$$\eta^{AB}\xi_{AB} = 0, \quad \xi_{AB}p^B = \xi_{BA}p^B = \xi_A, \quad \xi_A p^A = 0. \tag{2.8.43}$$

Notice that some of the states (2.8.41) and (2.8.42) have negative norm. Their presence is to be expected, since we are dealing with a model possessing six time-like coordinates. In this case, the Virasoro constraints (see eq. (2.8.45) below) cannot, by themselves alone, eliminate all ghosts. The same remarks given previously apply to these ghost states: their mass is always $O(M_{Pl})$.

All elements of $\mathbf{g}$ are line integrals of primary fields of conformal weight one of the Virasoro algebra generated by

$$L(z) = -\frac{1}{2}\eta_{AB}\partial X^A \partial X^B(z). \tag{2.8.44}$$

This statement is easily proven, since it holds true for all states given in (2.8.37) and (2.8.38), and the Jacobi identity together with $[L_n, A] = [L_n, B] = 0$ imply

$$[L_n, [A, B]] = [[L_n, A], B] - [[L_n, B], A] = 0, \quad n \geq 0. \tag{2.8.45}$$

To define an invariant tensor η^{ST} on all elements of $\mathbf{g}$ is equally easy. Indeed, by writing the elements of $\mathbf{g}$ as

$$V^S = \oint \frac{dz}{2\pi i}\phi^S(z), \tag{2.8.46}$$

where $\phi^S(z)$ is a primary field of conformal dimension one, and denoting with $|0\rangle$ the $SL(2, \mathbf{C})$ invariant vacuum of (2.8.44), we may define η^{ST} as

$$\eta^{ST} = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2}\langle 0|\phi^S(-1/\epsilon)\phi^T(\epsilon)|0\rangle. \tag{2.8.47}$$

This definition is nothing but the BPZ scalar product [26]. Hermiticity of η^{ST} is therefore immediate [26]. The invariance of (2.8.47) is proven by using representation (2.8.46) for the elements of $\mathbf{g}$, and by use of the Cauchy theorem

$$f^V_{ST}\eta_{VU} + f^V_{SU}\eta_{TV} = \lim_{\epsilon \to 0}\langle 0| \oint_C \frac{dz}{2\pi i}\phi_S(z)\frac{1}{\epsilon^2}\phi_T(-1/\epsilon)\phi_U(\epsilon)|0\rangle = 0, \tag{2.8.48}$$

where C is a contour encircling both points ϵ and $-1/\epsilon$.

As we recalled earlier, the invariant metric η_{ST} has more than six negative eigenvalues. This simply signals the limits of validity of the LEEA approach. Gauged $N = 4$ supergravity can be constructed for any invariant tensor obeying eq. (2.8.39) and of signature $(6+n, m)$ [74, 75, 28, 279]. In this case the $N = 4$ matter scalars live on the coset manifold

$$\frac{O(6+n, m, \mathbf{R})}{O(6, \mathbf{R}) \times O(n, m, \mathbf{R})}, \tag{2.8.49}$$

and as expected there are scalar ghosts propagating[19].

The fundamental property of the DISG algebra $\mathbf{g}$ is that its automorphism group contains the duality group $O(6,22,\mathbf{Z})$, as mentioned before. Since all elements of $\mathbf{g}$ can be written as commutators of elements (2.8.37) and (2.8.38), we need only to show that $O(6,22,\mathbf{Z})$ acts as an algebra isomorphism on these elements. Let us define the action of $O(6,22,\mathbf{Z})$ on $\mathbf{g}$ by

$$X^A(z) \to \Lambda^A_B X^B(z), \quad p^A \to (\Lambda^{-1})^A_B p^B, \quad \Lambda^A_B \in O(6,22,\mathbf{Z}). \tag{2.8.50}$$

By inspection of eqs. (2.8.37) and (2.8.38), $O(6,22,\mathbf{Z})$ is an isomorphism if

$$C(\Lambda^{-1}p) = \hat{\Lambda}^{-1} C(p) \hat{\Lambda}. \tag{2.8.51}$$

The linear operator $\hat{\Lambda}$ acts on the same Hilbert space on which $C(p)$ is defined. Property (2.8.51) is demonstrated in the appendix of ref. [144], where an explicit construction of $\hat{\Lambda}$, $\varepsilon(p,q)$, and $C(p)$ is given.

The DISG contains infinitely many affine Lie algebras. To see this, let us select a set of vectors p of $\Gamma^{(6,22)}$ of the form

$$p = (0,0,0,0,0,0,\alpha), \quad \alpha^2 = 2. \tag{2.8.52}$$

Let us choose the α so that they generate a finite dimensional Lie algebra, say $SO(44) \times U(1)^6$. Let us further select a vector $p_0 \in \Gamma^{(6,22)}$ of zero norm $p_0 \eta p_0 = 0$, and orthogonal to all α. Define now the generators

$$E^\alpha_n \equiv V^{\alpha + n p_0}, \quad H^A_n = \oint \frac{dz}{2\pi i} C(n p_0) \partial X^A \exp(i n p_{0A} X^A)(z). \tag{2.8.53}$$

The commutators of these generators read

$$\begin{aligned} [E^\alpha_n, E^\beta_m] &= \begin{cases} \varepsilon(\alpha,\beta) E^{\alpha+\beta}_{n+m} & \alpha\cdot\beta = -1 \\ \alpha_A H^A_{m+n} + m\delta_{m,-n} p_0 \eta H_0 & \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \\ [H^A_n, E^\alpha_m] &= \alpha^A E^\alpha_{n+m}, \qquad [H^A_n, H^B_m] = m\delta^{AB}\delta_{m,-n} p_0 \eta H_0. \end{aligned} \tag{2.8.54}$$

Here we made use of the equation

$$p_{0A} H^A_m = \oint \frac{dz}{2\pi i} p_{0A} \partial X^A \exp(i m p_{0A} X^A)(z) = 0, \quad \text{if } m \neq 0. \tag{2.8.55}$$

This identity holds because the integrand can be written as a total derivative. It is immediate to recognize in eq. (2.8.54) the commutation relations of an affine Lie algebra of level $p_0 \eta H$ [180, 229, 21].

[19] Together with their supersymmetric partners.

To conclude this section, let us notice that there is a way of avoiding ghosts in an effective action which includes massive fields. One can indeed pick up twenty-eight linearly independent positive lattice vectors p_i, $i = 1, .., 28$, of Lorentzian length two and such that $p_i \eta p_j \geq 0$ and form with them a (hyperbolic) Kac-Moody algebra. The procedure is standard [180]. To any length-two vector $\pm p_i$, one associates a generator $V_{\pm p_i}$. One then considers the algebra $\mathcal{G}$ freely generated by the $V_{\pm p_i}$, the vertices V^A, $A = 1, .., 28$, and the commutation relations

$$
\begin{aligned}
[V_{p_i}, V_{-p_j}] &= \begin{cases} \varepsilon(p_i, -p_j) V_{p_i - p_j} & p_i \eta p_j = 1 \\ p_{iA} V^A & i = j \\ 0 & \text{otherwise} \end{cases} \\
[V_{p_i}, V_{p_j}] = 0, &\qquad [V^A, V_{\pm p_i}] = \pm p_i^A V_{\pm p_i}, \\
[V^A, V^B] = 0, &\qquad A, B = 1, .., 28.
\end{aligned}
\tag{2.8.56}
$$

By dividing $\mathcal{G}$ by the (unique) maximal ideal commuting with all V^A one defines a hyperbolic Kac-Moody algebra $\hat{\mathcal{G}}$ [180, 229].

Among the known properties of $\hat{\mathcal{G}}$ there is the following: there exists an invariant Hermitian form η_{ST}, reducing to η_{AB} on the Cartan subalgebra, and positive definite outside it [180]. In our case this means that there exists an η with only six negative eigenvalues. Thus, the algebra $\hat{\mathcal{G}}$ gives rise to a ghost-free $N = 4$ supergravity [74, 75, 279]. The problem is that this theory is invariant only under a subgroup of $O(6, 22, \mathbf{Z})$ [20]. We think that complete duality invariance is more important than the presence of non-physical ultramassive ghosts, which, anyway, become relevant only at energies so high as to render the LEEA approach inapplicable. For this reason we suggest that the relevant LEEA gauge algebra is the algebra of the DISG, i.e. $\mathbf{g}$.

Finally, we comment about few relations with Closed-String Field Theory (CSFT).

In ref. [204] it was shown that $O(d, d, \mathbf{Z})$ is a symmetry of CSFT of the bosonic string expanded around flat backgrounds, as is suggested by the discussion of section 2.6. Moreover, in ref. [138] it was shown that a truncation of the string field gives rise to an effective action compatible to cubic order with the duality invariant LEEA for any choice of a cyclic string vertex. From CSFT one may learn how to add the $p\eta p = 0$ modes to the effective action.

As we mentioned before, to construct the complete invariance algebra of the string, we must add higher spin states to the algebra. To bypass this difficulty it is easier to study first the gauge symmetries of the $N = 2$ string, as the higher spin modes are never physical in that case. This was done in ref. [149]. A suggestion for the complete invariance algebra of the bosonic string was put forward along the lines of this section in ref. [230].

We conclude here our study of the $D = 4$, $N = 4$ duality-invariant effective Lagrangian of the heterotic string. In the next section we shall extend the analysis to more general compactifi-

[20] This fact can be easily proven by noticing that any root of $\hat{\mathcal{G}}$ is either positive or negative [180], and thus $\hat{\mathcal{G}}$ cannot contain all length two vectors of $\Gamma^{(6,22)}$.

cations. In particular, we shall study string theory on orbifolds and Calabi-Yau manifolds from the effective-action perspective and, briefly, from the (microscopic) worldsheet viewpoint.

3 Duality and Discrete Symmetries of the Moduli Space of Orbifold and Calabi-Yau Backgrounds

The previous section dealt with the case of toroidal compactifications of the heterotic string. In order to make contact with phenomenology, on the other hand, strings must be compactified on more complicated background as, for instance, orbifolds [86] or Calabi-Yau spaces [42]. In these cases one may also define duality between seemingly different backgrounds.

Before discussing more complicated cases let us mention the simplest example of non-flat compactification and its duality symmetry, namely the "orbicircle" example.

The orbicircle is defined by identifying the points x and $-x$ on the circle $x \approx x + 2\pi R m$, $m \in \mathbf{Z}$ (see figure 1.D). This identification gives rise to a closed segment whose two boundary points correspond to the fixed points under $x \to -x : \ x = 0$ and $x = \pi R$. Even in this case there is a duality transformation relating a length πR orbicircle with another one of length π/R. This can be shown at the level of the spectrum [94] of the corresponding CFT along the lines of section 2.2.

The orbicircle compactifications span a sub-space of $c = 1$ backgrounds. The moduli space of $c = 1$ unitary CFTs is shown in figure 3.A [22, 80, 133, 195]. It consists of target spaces which are circles of radius R_c, target spaces which are orbicircles of radius R_o, and three special orbifolds [133]. These three points correspond to CFTs without marginal deformations. The local dimension of the moduli space is zero at those points. Along the R_c- and R_o-line the dimension is one. There is a special point where the two lines meet. At this point, a string cannot distinguish if it moves on the circle or on the orbicircle. This point is the self-dual point of the orbifold, with a symmetry enhanced from $\mathbf{Z}_2 \times \mathbf{Z}_2$ to $U(1)_L \times U(1)_R$.

We return now to compactifications to 4-D, in which case no complete classification is presently available. A few particular examples of orbifold compactifications are studied in some detail in section 3.1. The subject has been studied in more general or more complex situations in several papers as, for instance, in refs. [207, 239, 209, 228, 291, 95, 97, 96, 99, 265, 266, 100, 98, 267, 17].

In section 3.2 we extend the analysis of sections 2.7 and 2.8 to the $N = 1$ and $N = 2$ orbifold case. Namely, we find 4-D effective actions invariant under the complete duality group or subgroups thereof. $N = 1$ orbifolds are particular (singular) examples of Calabi-Yau spaces. The effective actions for those compactifications are conveniently described using the language of "special geometry," which we introduce in section 3.3. The study of duality in Calabi-Yau spaces is briefly surveyed in section 3.4. In the same section we also review various results concerning the structure of the superpotential of $N = 1$ compactifications; it turns out that duality significantly modifies the naively expected low energy physics.

3.1 Duality in Orbifold Compactifications

Reducing $N = 4$ to $N = 2, 1, 0$ supersymmetry in four dimensions can be achieved in a simple way by means of an orbifold construction [86]. A (symmetric) orbifold construction based on T^6 (the six-dimensional, compact, internal manifold generating $\Gamma^{(6,6)}$), involves the following steps:

1. Identify points of T^6 conjugated by a group element $g \in G$. The group G is a symmetry of T^6, e.g. a discrete rotation. In the following, for sake of simplicity, we shall take $G = \mathbf{Z}_N$. This choice gives rise to an Abelian orbifold [86]. In general, the action of $\mathbf{Z}_N$ on T^6 is not free. In other words, there exist points of T^6 which are invariant under $\mathbf{Z}_N$. For instance, as mentioned above, the one dimensional torus T^1 (namely, a radius R circle) can be modded out by the $\mathbf{Z}_2$ symmetry $x \to -x$. This symmetry has two fixed points: $x = 0$ and $x = \pi R$. The existence of fixed points means that the quotient space $T^6/\mathbf{Z}_N$ is not a smooth manifold, but rather possesses conical singularities. In spite of this fact, the resulting string theory is consistent, unlike the corresponding Kaluza-Klein field theory.

   Once $\mathbf{Z}_N$ has been defined on the compactified coordinate, one defines its action on the gauge indices. In the $\mathbf{Z}_2$ example, for the $E_8 \times E_8$ heterotic string, one may decompose one of the E_8 factors of the gauge group into representations of the maximal E_8-subgroup, $SU(2) \times E_7$. Then, one may identify the $\mathbf{Z}_2$ acting on the torus coordinate with the center of $SU(2)$.

2. Project the Hilbert space $\mathcal{H}$ of the $N = 4$ heterotic string onto $\mathbf{Z}_N$ invariant states. Explicitly, the projection reads
$$P = \frac{1}{N} \sum_{m=1}^{N} g^m. \tag{3.1.1}$$
   Here g is the generator of $\mathbf{Z}_N$, obeying $g^N = 1$.

3. Unlike the field-theoretical case, this is not the end of the story. Indeed, one must also take into account those string configurations that close only up to a $\mathbf{Z}_N$ transformation:
$$X^A(\sigma + 2\pi, \tau) = (hX)^A(\sigma, \tau), \quad h \in \mathbf{Z}_N. \tag{3.1.2}$$
   These configurations generate new string states. The Hilbert space they span is called the *twisted sector*.

The $N = 1$ or $N = 2$ LEEA of an orbifold compactification contains two types of fields: the untwisted ones, which form a subset of the $N = 4$ fields, and the twisted fields. The twisted fields are new, additional fields, not present in the truncation of the $N = 4$ LEEA. Their Lagrangian, therefore, is not fixed by supersymmetry and field-theoretical arguments alone [21].

[21] A remarkable exception is found in the fermionic construction of four-dimensional strings [10, 189]. In that case one can figure out a $\mathbf{Z}_2$ truncation of a four-dimensional $N = 2$ theory which breaks supersymmetry to $N = 1$ without introducing additional massless states. The LEEA for the massless states of the resulting $N = 1$ theory is determined by supersymmetry and a few other properties [104].

In this section we find out what subgroup of the $O(6,6,\mathbf{Z})$ duality group survives the orbifold truncation, and find the transformation laws of untwisted *and twisted* massless states under such duality. The more ambitious problem of finding a LEEA, invariant under the $O(6,22,\mathbf{Z})$ subgroup surviving the orbifold truncation, is (partially) answered in section 3.2.

The orbifold construction sketched above can be extended to a general $\Gamma^{(6,22)}$ compactification [237]. Here we deal, for simplicity, with $\Gamma^{(6,6)} \oplus \Gamma^{(0,16)}$ (symmetric) orbifolds.

Let us define explicitly the action of $\mathbf{Z}_N$ on the string coordinates. To keep the notation simple we will consider the $\mathbf{Z}_3$ orbifold, giving rise to $N = 1$ supersymmetric theory. The generalization to any $\mathbf{Z}_N$ is done in [86].

First of all, by complexifying the six internal torus coordinates we may write

$$(gX)^i(z,\bar{z}) = e^{2\pi i/3} X^i(z,\bar{z}), \quad X^{\bar{\imath}} \equiv (X^i)^*, \quad i = 1,2,3, \tag{3.1.3}$$

namely, from now on we work with the complex torus $T^6 = (T^2)^3$. Notice that we choose a $\mathbf{Z}_3$ acting in the same way on all coordinates. By denoting with $X(\bar{z})^A$, $A = 1, ..., 16$, the 16 chiral left-moving bosons taking value in the maximal torus of $E_8 \times E_8$ (or $Spin(32)/\mathbf{Z}_2$), we further define

$$(gX)^A(\bar{z}) = X^A(\bar{z}) + 2\pi\delta^A. \tag{3.1.4}$$

The shift δ^A obeys $3\sum_{A=1}^{16} \delta^A p^A \in \mathbf{Z}$ for all p^A in $\Gamma^{(0,16)}$.

Equations (3.1.3, 3.1.4) induce the following transformation on the $\Gamma^{(6,6)}$ vector $(k_L^i,\, k_R^i)$

$$(gk)_L^i = e^{2\pi i/3} k_L^i, \quad (gk)_R^i = e^{2\pi i/3} k_R^i. \tag{3.1.5}$$

The $k_{L,R}^i$ are expressed in terms of the momenta and winding numbers of X^i by eqs. (2.5.14, 2.5.15).

The gauge algebra is realized in terms of $(1,0)$ conformal vertices, as explained in section 2.8 for the DISG algebra. Here they take the form

$$\begin{aligned} V_p &= C(p) \oint \frac{d\bar{z}}{2\pi i} e^{i\sum_A X^A(\bar{z}) p^A}, \quad \sum_1^{16} p^A p^A = 2, \quad p^A \in \Gamma^{(0,16)}, \\ V_A &= \oint \frac{d\bar{z}}{2\pi i} \bar{\partial} X^A(\bar{z}). \end{aligned} \tag{3.1.6}$$

The cocycle $C(p)$, in this case, can be written down explicitly as follows [162]. Pick up a basis $\{e_M | M = 1, ..., 16\}$ of $\Gamma^{(0,16)}$, and choose the e_M such that $e \cdot e \equiv \sum_A e_M^A e_M^A = 2$. Then the cocycle reads

$$C(p) \equiv (-)^{p * \hat{p}}, \quad X * Y \equiv \sum_{M>N} X^M Y^N e_M \cdot e_N. \tag{3.1.7}$$

Here $\hat{p}$ is the momentum operator of the left-moving chiral bosons.

The $\mathbf{Z}_3$ action on vertices (3.1.6) reads [86]

$$(gV)_p = e^{2\pi i \sum_A \delta^A p^A} U V_p U^\dagger, \quad (gV_A) = V_A, \tag{3.1.8}$$

where U is a unitary transformation acting on the cocycle $C(p)$ only. This transformation simply defines an equivalent representation of the cocycle, as noticed in section 2.8.

Let us examine now the behavior of the orbifold compactification under duality, beginning by analysing the states in the untwisted sector.

Equations (3.1.3, 3.1.4) completely specify the projection (3.1.1) on all untwisted states. These states can be written as a product of an oscillator part and a $\Gamma^{(6,6)}$-lattice part. It is useful to decompose the lattice states in the following way

$$|k\rangle_m = \sum_{n=1}^{3} e^{2\pi i m n/3} |e^{2\pi i n/3} k\rangle. \tag{3.1.9}$$

This linear superposition of lattice vectors transforms as

$$g|k\rangle_m = e^{-2\pi i m/3} |k\rangle_m. \tag{3.1.10}$$

If we denote by O_m a polynomial in the oscillators of the string coordinates, transforming as

$$(gO_m) = e^{2\pi i m/3} O_m, \tag{3.1.11}$$

the invariant states are simply $O_m|k\rangle_{m'}$, with $m - m' = 0 \bmod 3$. Notice that at $k = 0$ the only invariant states read $O_0|0\rangle$.

The subgroup of $O(6,6,\mathbf{Z})$ surviving the orbifold truncation can now be determined quite simply.

Let us find at first what constraints follow from the lattice-dependent part of the vertex $O_m|k\rangle_{m'}$. By definition, a symmetry transforms physical states into physical states, so, the duality transform of $O_m|k\rangle_m$ must be $\mathbf{Z}_3$ invariant too. This requirement implies in particular that

$$g|\Lambda k\rangle_m = e^{-2\pi i m/3} |\Lambda k\rangle_m, \quad \Lambda \in O(6,6,\mathbf{Z}). \tag{3.1.12}$$

In general

$$(\Lambda k)^i = \Lambda^i_j k^j + \Lambda^i_{\bar{j}} k^{\bar{j}}, \tag{3.1.13}$$

thus, the Λ satisfying eq. (3.1.12) obey

$$\Lambda^i_{\bar{j}} = \Lambda^{\bar{i}}_j = 0. \tag{3.1.14}$$

This equation is quite natural, it says that the surviving duality group is included in $O(6,6,\mathbf{Z}) \cap SU(3,3,\mathbf{R})$ [110, 103], namely, the duality subgroup preserving the complex structure of $(T^2)^3$.

This group, called $SU(3,3,\mathbf{Z})$, commutes with the orbifold projection and, therefore, transforms O_m, that is the oscillator dependent part of a physical vertex, into another operator O'_m which transforms under $\mathbf{Z}_3$ as O_m. This means that $SU(3,3,\mathbf{Z})$ maps massless physical states into themselves. Invariance under this group puts severe constraints on the form of the $N=1$ LEEA arising from this orbifold compactification.

We have not yet shown that $SU(3,3,\mathbf{Z})$ is an invariance of the twisted sector too. Here we shall give a sketch of this proof, carried on in detail in ref. [208] [22].

In the twisted sector, the string coordinates obey the boundary conditions

$$\begin{aligned} X^i(\tau,\sigma+2\pi) &= e^{2\pi i m/3} X^i(\tau,\sigma), \quad \psi^i_R(\tau,\sigma+2\pi) = e^{2\pi i m/3}\psi^i_R(\tau,\sigma), \\ X^A_L(\tau+\sigma+2\pi) &= X^A_L(\tau+\sigma) + 2\pi m \delta^A, \qquad m=1,2. \end{aligned} \tag{3.1.15}$$

The boundary conditions of the Neveu-Schwarz fermions ψ^i_R are dictated by worldsheet supersymmetry. The boundary conditions in (3.1.15) entail the following mode expansion

$$\begin{aligned} X^i(\tau,\sigma) &= f^i_a + \frac{i}{\sqrt{2}} \sum_{n\in\mathbf{Z}+m/3} \frac{1}{n}\alpha^i_n e^{-in(\tau-\sigma)} + \frac{i}{\sqrt{2}} \sum_{n\in\mathbf{Z}-m/3} \frac{1}{n}\tilde{\alpha}^i_n e^{-in(\tau+\sigma)}, \\ \psi^i_R(\tau,\sigma) &= \sum_{n\in\mathbf{Z}+m/3+1/2-t/2} \psi^i_n e^{-in(\tau-\sigma)}, \\ X^A_L(\tau+\sigma) &= (p^A + m\delta^A)(\sigma+\tau) + \frac{i}{\sqrt{2}} \sum_{n\in\mathbf{Z}} \frac{1}{n}\tilde{\alpha}^A_n e^{-in(\tau+\sigma)}, \quad p^A \in \Gamma^{(0,16)}. \end{aligned} \tag{3.1.16}$$

In this equation f^i_a denote the T^6 fixed points of the $\mathbf{Z}_3$ transformation $g^m \equiv \exp(2\pi i m/3)$. The index a labels these fixed points and the integer t is equal to 0 in the Neveu-Schwarz sector, and to 1 in the Ramond sector. Notice that the mode expansion of the heterotic chiral boson X^A is unchanged, but for a translation by $m\delta^A$ of the lattice vector p^A. There is a mode expansion given by eq. (3.1.16), and therefore, a Hilbert space of states for each one of the fixed points f_a. In other words, each twisted sector (with twist g^m) is made of many identical copies, one for each fixed point.

By denoting with O_k, as before, a polynomial in the string oscillators obeying $(gO)_k = \exp(2\pi i k/3)$, and by $|p^A, a\rangle$ a state at the a-th fixed point with heterotic momentum p^A, we may write a $\mathbf{Z}_3$ invariant twisted state as

$$O_k |p^A + m\delta^A, a\rangle, \quad k + \delta^A(p^A + m\delta^A) = 0 \bmod 3. \tag{3.1.17}$$

Notice that the twisted states have $k^i = k^{\bar{i}} = 0$, since a non-zero momentum or winding number in T^6 is incompatible with boundary conditions (3.1.15).

[22] Ref. [208] deals with invariance under an $SL(2,\mathbf{Z})$ subgroup of $SU(3,3,\mathbf{Z})$, but its techniques can be extended to the general case.

It may seem that the state (3.1.17) be mapped into itself by a duality transformation, since it does not depend on k^i, $k^{\bar{i}}$. This is not true, however. States, of identical oscillator and momentum content, belonging to different fixed points, are isomorphic (they have the same conformal weight etc.) and can mix together. In general, under duality states (3.1.17) transform as [208]

$$O_k|p^A + m\delta^A, a\rangle \to e^{i\varphi(G,B,\Lambda)} S_{ab}(\Lambda) O_k|p^A + m\delta^A, b\rangle, \quad \Lambda \in SU(3,3,\mathbf{Z}). \tag{3.1.18}$$

The phase $\varphi(G,B,\Lambda)$ depends on the background fields G, B, but not on the fixed point, whereas the unitary matrix $S_{ab}(\Lambda)$, transforming different fixed points into each-other, is independent of the background.

Equation (3.1.18) is sufficient to conclude that the spectrum of an orbifold compactification is invariant under duality. Indeed, eq. (3.1.18) maps physical states into physical states, and defines a unitary transformation between equivalent Hilbert spaces.

The explicit form of S_{ab} and φ was given in ref. [208]. There it was also shown that not only the twisted-sector spectrum, but also its interactions, namely the three- and four-point correlation functions, are invariant under duality, if the twisted vertices are transformed as in eq. (3.1.18).

The previous analysis has dealt with a simple question, namely, what is the duality group of an orbifold compactification. The next question one is naturally led to is how this duality acts on fields, in particular on the massless ones. To answer this question one must find how twisted and untwisted vertices depend on the background. Here, for sake of simplicity, we choose T^6 to be a product of three *identical* two dimensional tori T^2. The four T^2-moduli (background fields) B_{ij}, G_{ij}, $i,j = 1,2$ reduce to two, once the complex structure has been defined by

$$X(z,\bar{z}) = X^1(z,\bar{z}) + e^{2\pi i/3} X^2(z,\bar{z}). \tag{3.1.19}$$

This identification leaves one free complex modulus, since the metric of a complex 2-d manifold has only one independent component. Let us define this modulus by

$$\tau = i\sqrt{\det G} + B_{12}. \tag{3.1.20}$$

Then, our general formulæ (2.5.14, 2.5.15) giving p_L and p_R in terms of the background fields reads

$$\begin{aligned} p_L &= \frac{i}{(\sqrt{3}\Im\tau)^{1/2}}[m_2 + e^{-2\pi i/3} m_1 + \bar{\tau}(n_1 - e^{-2\pi i/3} n_2)], \\ p_R &= \frac{i}{(\sqrt{3}\Im\tau)^{1/2}}[m_2 + e^{-2\pi i/3} m_1 + \tau(n_1 - e^{-2\pi i/3} n_2)]. \end{aligned} \tag{3.1.21}$$

Here $m_i, n_i \in \mathbf{Z}$.

There are three moduli τ_i, one for each T^2 factor in $T^6 = T^2 \otimes T^2 \otimes T^2$. The subgroup of $SU(3,3,\mathbf{Z})$ most frequently studied in the literature is $SL(2,\mathbf{Z})^3$. This subgroup acts on the τ_i as follows

$$\tau_i \to \frac{a_i\tau_i + b_i}{c_i\tau_i + d_i}, \quad a_i, b_i, c_i, d_i \in \mathbf{Z},\ a_i d_i - b_i c_i = 1. \tag{3.1.22}$$

We notice that the group acts projectively on the moduli, thus, it should more properly be called $PSL(2,\mathbf{Z})^3$.

Equation (3.1.22) induces a $SL(2,\mathbf{Z})^3$ transformation on the lattice (3.1.21). This transformation can be realized by acting on the fields $X_R^i(z)$, $X_L^i(\bar{z})$ as follows

$$X_L^i(\bar{z}) \to \bar{\lambda}\left(\frac{c_i\tau_i + d_i}{c_i\bar{\tau}_i + d_i}\right)^{1/2} X_L^i(\bar{z}), \quad X_R^i(z) \to \lambda\left(\frac{c_i\bar{\tau}_i + d_i}{c_i\tau_i + d_i}\right)^{1/2} X_R^i(z). \tag{3.1.23}$$

Obviously, as dictated by worldsheet supersymmetry, the right-moving fermions $\psi_R(z)$ transform as the coordinates

$$\psi_R^i(z) \to \lambda\left(\frac{c_i\bar{\tau}_i + d_i}{c_i\tau_i + d_i}\right)^{1/2} \psi_R^i(z). \tag{3.1.24}$$

In these equations, λ is a $SL(2,\mathbf{Z})$-dependent, but background independent phase [208, 178, 176].

Let us find how $SL(2,\mathbf{Z})^3$ acts on the vertices of massless untwisted scalars. The transformation law on target space fermions has been presented in ref. [178, 176], and follows by imposing target space supersymmetry on the scalars. The scalar vertices read

$$\begin{aligned} \phi_{(\bar{3},\bar{27})} &= \int d^2z \psi_R^i(z) V_L^{(\bar{3},\bar{27})}(\bar{z}), \quad \phi_{(3,27)} = \int d^2z \psi_R^{\bar{i}}(z) V_L^{(3,27)}(\bar{z}), \\ \phi^{i\bar{j}} &= \int d^2z \psi_R^i(z) \bar{\partial} X_L^{\bar{j}}(\bar{z}). \end{aligned} \tag{3.1.25}$$

Here $V_L^{(a,b)}(\bar{z})$ denotes a $E_8 \times E_8$ vertex, given by eq. (3.1.6), transforming in the (a,b) representation of $SU(3) \times E_6$. The breakdown $E_8 \to SU(3) \times E_6$ is achieved by setting the lattice shift δ^A equal to $(1/3, 1/3, 1/3, 0, ..., 0)$.

Equations (3.1.23, 3.1.24) give the following duality transformation laws for the massless, untwisted scalar vertices

$$\begin{aligned} \phi_{(\bar{3},\bar{27})} &\to \prod_i \lambda\left(\frac{c_i\bar{\tau}_i + d_i}{c_i\tau_i + d_i}\right)^{1/2} \phi_{(\bar{3},\bar{27})}, \quad \phi_{(3,27)} \to \prod_i \bar{\lambda}\left(\frac{c_i\tau_i + d_i}{c_i\bar{\tau}_i + d_i}\right)^{1/2} \phi_{(3,27)}, \\ \phi^{i\bar{j}} &\to \prod_i \left(\frac{c_i\bar{\tau}_i + d_i}{c_i\tau_i + d_i}\right) \phi^{i\bar{j}}. \end{aligned} \tag{3.1.26}$$

The transformation laws of the twisted vertices are more involved, since states corresponding to different sectors transform into each other. Nevertheless, one may fix the τ_i-dependent part of the duality transformations by a few simple considerations.

At first, one notices that transformations (3.1.23, 3.1.24) act as chiral rotations on the string coordinates. The corresponding chiral charges are determined by the mode expansion of $X^i_{L,R}$ and ψ^i_R in the m-twisted sector, given in eq. (3.1.16)

$$\begin{aligned} Q^i_F &= \sum_{n\in\mathbf{Z}+m/3+1/2-t/2} \bar\psi^{\bar\imath}_{-n}\psi^i_n - m/3 - t/2 \\ Q^i_{B,L} &= \sum_{n\in\mathbf{Z}-m/3} \frac{1}{n}\tilde\alpha^{\bar\imath}_{-n}\tilde\alpha^i_n + m/3, \quad Q^i_{B,R} = \sum_{n\in\mathbf{Z}+m/3} \frac{1}{n}\alpha^{\bar\imath}_{-n}\alpha^i_n - m/3. \end{aligned} \tag{3.1.27}$$

The shift in the ground-state chiral charge is dictated by spectral flow [31].

Let us analyse the $m = 1$ twisted sector. (The $m = 2$ case can be worked out in complete analogy with the previous one).

In this sector, the massless scalar states are [86]

$$|p^A + \delta^A, a\rangle_L \otimes |0, a\rangle_R, \tag{3.1.28}$$

and

$$\tilde\alpha^{\bar\imath}_{-1/3}|p^A + \delta^A, a\rangle_L \otimes |0, a\rangle_R. \tag{3.1.29}$$

The ground state of the left-movers $|p^A + \delta^A\rangle$ contains a non-zero vector in the shifted $E_8 \times E_8$ lattice. This is a vector in the weight lattice of $SU(3)\times E_6\times E_8$. In eq. (3.1.28) this vector spans the $(1, 27, 0)$ representation of $SU(3) \times E_6 \times E_8$, while in eq. (3.1.29) it spans the $(3, 1, 0)$ [86].

The duality transformations (3.1.23, 3.1.24) are given by $\exp i\sum_{i=1}^3 (Q^i_R - Q^i_L)\theta_i$ with $2\theta_i = \log[(c_i\tau_i+d_i)/(c_i\bar\tau_i+d_i)]$. By denoting with $\phi^a_{(1,27)}$ and $\phi^{ia}_{(3,1)}$ the vertices associated to states (3.1.28) and (3.1.29), respectively, and using the formula for the chiral charges given in eq (3.1.27), one finds that $SL(2,\mathbf{Z})^3$ acts on the twisted states as follows

$$\begin{aligned} \phi^a_{(1,27)} &\to \prod_i \left(\frac{c_i\bar\tau_i + d_i}{c_i\tau_i + d_i}\right)^{1/3} S^1_{ab}\phi^b_{(1,27)}, \\ \phi^{ia}_{(3,1)} &\to \left(\frac{c_i\bar\tau_i + d_i}{c_i\tau_i + d_i}\right)^{5/6} \left(\frac{c_j\bar\tau_j + d_j}{c_j\tau_j + d_j}\right)^{1/3} \left(\frac{c_k\bar\tau_k + d_k}{c_k\tau_k + d_k}\right)^{1/3} S^2_{ab}\phi^{ib}_{(3,1)}, \quad i\neq j\neq k. \end{aligned} \tag{3.1.30}$$

As in eq. (3.1.18), vertices corresponding to different fixed points transform into each other. The matrices $S^1_{(ab)}$, $S^2_{(ab)}$, describing this mixing, depend on a_i, b_i, c_i, d_i, *but are independent of the modular parameter* τ_i. These matrices cannot be determined by our simple method, they are given explicitly in ref. [208].

Equations (3.1.26, 3.1.30) completely fix the moduli-dependent part of the $SL(2,\mathbf{Z})^3$ duality on all massless scalars of the $\mathbf{Z}_3$ orbifold compactification. These formulas can be easily extended to any $\mathbf{Z}_N$ orbifold.

As we previously stated, the LEEA containing massless states must be exactly invariant under $SU(3, 3, \mathbf{Z})$, and thus under $SL(2,\mathbf{Z})^3$. This property, together with the explicit form of

duality on massless fields, imposes powerful constraints on the form of the LEEA. Some of the consequences of this constraint will be studied in section 3.4.

3.2 Duality Invariant Effective Field Theories of $N = 2, 1$ Compactifications

In this section, following [144], duality in $N = 1, 2$, compactifications will be analyzed. We study the "large" duality interchanging massive and massless modes. This is the symmetry that places most restrictions on the LEEA. Unfortunately, unlike the $N = 4$ case, for $N = 2, 1$, supersymmetry arguments alone are not strong enough to fix the form of the LEEA completely. The presence of additional twisted sectors in the orbifold construction, for instance, makes the knowledge of the $N = 4$ LEEA insufficient to determine the complete low-energy $N = 2, 1$ actions. The $N = 4$ LEEA is, on the other hand, all what we need to find the $N = 2, 1$ effective actions for the untwisted states. The derivation of such LEEA's is the subject of this section. In section 3.4, we shall examine the restrictions imposed on orbifold compactifications by the "small" duality $SL(2, \mathbf{Z})$, transforming massless states into themselves. In this case the analysis can be extended, and it gives constraints on the twisted sector as well.

To study the LEEA of the untwisted sector of orbifold compactifications, we need some definitions, which extend those of the previous section.

A $(\mathbf{Z}_N)$ orbifold projection is defined by picking an element $\mathcal{O}$ of $O(6, \mathbf{Z}) \times O(6, \mathbf{Z}) \subset O(6, 22, \mathbf{Z})$. For simplicity, we consider here the case of an $\mathcal{O}$ acting symmetrically on left and right movers, even though this restriction is by no means necessary [237]. Moreover, we choose $\mathcal{O}$ such that $\mathcal{O}^N = 1$ for some integer $N < \infty$. The action of $\mathcal{O}$ on the LEEA fields is defined by specifying how $\mathcal{O}$ acts on the gauge indices as well as the $N = 4$ "extension" indices. For the gauge indices, one defines how $\mathcal{O}$ acts on the chiral fields $X^A(z)$ of eq. (2.8.36)

$$\begin{aligned} \mathcal{O}X^A(z) &= \Lambda^A_B X^B(z) + 2\pi\delta^A, \quad \Lambda^A_B = \mathrm{diag}(\lambda^a_b, \lambda^a_b, 0), \\ \lambda &\in O(6, \mathbf{Z}), \quad A, B = 1, .., 28, \quad a, b = 1, .., 6. \end{aligned} \tag{3.2.1}$$

The shift vector δ^A obeys

$$\delta^A = 0, \quad A = 1, .., 12, \quad N p_A \delta^A \in \mathbf{Z}, \quad \forall p_A \in \Gamma^{(6,22)}. \tag{3.2.2}$$

Equations (3.2.1, 3.2.2) define $\mathcal{O}$ on all vertices. For example, since λ^a_b is an element of the duality group,

$$\mathcal{O}V^p = \oint \frac{dz}{2\pi i} C(p) \exp[i p_B (\Lambda^B_A X^A + 2\pi\delta^B)](z) = \exp(2\pi i p\eta\delta) V^{\Lambda^{-1} p}, \tag{3.2.3}$$

where V^p is defined in eq. (2.8.37), and the second equality in (3.2.3) is modulo a conjugation of $C(p)$.

One can further define the action of $\mathcal{O}$ on the $N=4$ indices by specifying how it acts on the fundamental representation of $SU(4)$. This is because $\lambda \in SO(6)$, $SO(6) \simeq SU(4)$, and because the vectorial representation of $SO(6)$ is isomorphic to the antisymmetric of $SU(4)$. With an appropriate choice of basis, and by labelling with $|j\rangle$ the fundamental representation of $SU(4)$, we get

$$\begin{aligned} \lambda &= \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4), \quad m_j \in \mathbf{Z}, \quad \sum_{j=1}^{4} m_j = 0, \ \mathrm{mod}\ N. \\ \mathcal{O}|j\rangle &= \lambda|j\rangle = \lambda_j|j\rangle = \exp(2\pi i m_j/N)|j\rangle. \end{aligned} \tag{3.2.4}$$

By recalling that the $N=4$ matter multiplet of the LEEA reads

$$(A_\mu^S, \psi_i^S, Z_{ij}^S), \quad Z^{Sij} \equiv (Z_{ij}^S)^* = \frac{1}{2}\epsilon^{ijkl} Z_{kl}^S, \tag{3.2.5}$$

we define

$$\mathcal{O}(A_\mu^S, \psi_i^S, Z_{ij}^S)V^S \equiv (A_\mu^S, \exp(2\pi i m_i/N)\psi_i^S, \exp[2\pi i(m_i+m_j)/N]Z_{ij}^S). \tag{3.2.6}$$

This transformation is induced by the transformation of the vertex V^S given in eq. (2.8.46), namely, $(\mathcal{O}V)^S$.

The fields of the untwisted sector of the $N<4$ theories are those left invariant by the above transformation. If $\mathcal{O} \in SU(2) \subset SU(4)$, the resulting orbifold is $N=2$ space-time supersymmetric. If $\mathcal{O} \in SU(3)$, $\mathcal{O} \notin SU(2)$, the orbifold is $N=1$ supersymmetric. We turn now to a study of the LEEA for the untwisted sector [108, 106, 65, 112, 66].

A method for studying the untwisted sector of 4-D superstring effective actions has been proposed in refs. [108, 106, 112]. That method has been used in [112] to study systematically the scalar manifolds of $\mathbf{Z}_N$ orbifolds. The results of [112] can be generalized easily to the duality-invariant LEEA presented in section 2.8. Let us briefly review the methods of [108, 106, 112]. The fundamental fact used there is that the orbifold truncation acts on the compensating multiplet $(A_\mu^a, \psi_i^a, Z_{ij}^a)$ in a well defined way, given by eq. (3.2.6). For the compensating scalars one finds, by using eqs. (2.8.38), (3.2.1) and (3.2.6),

$$(\mathcal{O}Z)_{kl}^{ij} = \exp[2\pi i(m_k+m_l+m_i+m_j)/N]Z_{kl}^{ij}. \tag{3.2.7}$$

The compensators surviving the projection generated by $\mathcal{O}$, i.e. those left invariant by transformation (3.2.7), determine by themselves alone the geometry of the manifold of the (untwisted) scalar fields.

As in ref. [112] we find five different geometrical structures for $N=1$ theories, and two for $N=2$. Let us consider first the $N=1$ cases. The five different structures correspond to the following choice of the coefficients m_i in eq. (3.2.4) (all equalities below are modulo N, and

$m_4 = 0$)

$$
\begin{aligned}
&\text{a} \quad m_1 = m_2 = m_3 = N/3, \quad N \in 3\mathbf{Z}, \\
&\text{b} \quad m_1 = m_2 \neq m_3, \quad m_1 + m_2 + m_3 = 0, \quad m_i \neq N/2, \quad N \in 2\mathbf{Z}, \\
&\text{c} \quad m_1 \neq m_2 \neq m_3, \quad m_1 + m_2 + m_3 = 0, \quad m_i \neq N/2, \quad N \in 2\mathbf{Z}, \\
&\text{d} \quad m_1 = m_2 = N/4, \quad m_3 = N/2, \quad N \in 4\mathbf{Z}, \\
&\text{e} \quad m_1 \neq m_2, \quad m_3 = N/2, \quad m_1 + m_2 + m_3 = 0, \quad N \in 2\mathbf{Z}.
\end{aligned}
$$

The compensating scalars surviving the truncation defined by $\mathcal{O}$ are

$$
\begin{aligned}
Z^{4i}_{4j}, \quad i,j = 1,2,3 \qquad & \text{for case a} \\
Z^{4i}_{4j}, \quad i,j = 1,2, \quad Z^{43}_{43} \qquad & \text{for case b} \\
Z^{4i}_{4i}, \quad i = 1,2,3 \qquad & \text{for case c} \\
Z^{4i}_{4j}, \quad i,j = 1,2, \quad Z^{43}_{43}, \quad Z^{12}_{43} \qquad & \text{for case d} \\
Z^{4i}_{4i}, \quad i = 1,2, \quad Z^{43}_{43}, \quad Z^{12}_{43} \qquad & \text{for case e}
\end{aligned}
\tag{3.2.8}
$$

By substituting the fields of eq. (3.2.8) into the constraint (2.7.6), we find that the structure of the scalar manifold is always $SU(1,1)/U(1) \times \mathcal{K}$, where $\mathcal{K}$ is

$$
\begin{aligned}
&\frac{SU(3+n,m,\mathbf{R})}{U(1) \times SU(3,\mathbf{R}) \times SU(n,m,\mathbf{R})} \qquad \text{for case a} \\
&\frac{SU(2+n,m,\mathbf{R})}{U(1) \times SU(2,\mathbf{R}) \times SU(n,m,\mathbf{R})} \times \frac{SU(2+n',m',\mathbf{R})}{U(1) \times SU(2,\mathbf{R}) \times SU(n',m',\mathbf{R})} \qquad \text{for case b} \\
&\left[\frac{SU(1+n,m,\mathbf{R})}{U(1) \times SU(n,m,\mathbf{R})}\right]^3 \qquad \text{for case c} \\
&\frac{SU(2+n,m,\mathbf{R})}{U(1) \times SU(2,\mathbf{R}) \times SU(n,m,\mathbf{R})} \times \frac{O(2+n',m',\mathbf{R})}{O(2,\mathbf{R}) \times O(n',m',\mathbf{R})} \qquad \text{for case d} \\
&\left[\frac{SU(1+n,m,\mathbf{R})}{U(1) \times SU(n,m,\mathbf{R})}\right]^2 \times \frac{O(2+n',m',\mathbf{R})}{O(2,\mathbf{R}) \times O(n',m',\mathbf{R})} \qquad \text{for case e.}
\end{aligned}
\tag{3.2.9}
$$

These expressions are a little formal, since in our case $n = m = n' = m' = \infty$, but they can be made rigorous. Actually, some of these manifolds have been used in string theory also in [7].

To determine what matter fields actually survive the truncation defined by (3.2.6) we divide the group generators in two sets. The first one is made of the generators V^A, $A = 1, .., 28$. The corresponding scalar fields are the compensators, for which $\mathcal{O}$ has been defined in eq. (3.2.7), and the scalars Z^I_{ij}, $I = 7, .., 28$. By an appropriate choice of basis, since we consider only symmetric $\mathbf{Z}_N$ orbifolds, we may write

$$
Z^I_{ij} = (\tilde{Z}^{ab}_{ij}, Z^\alpha_{ij}), \quad \alpha = 13, .., 28, \quad i,j,a,b \in 4 \text{ of } SU(4),
\tag{3.2.10}
$$

and

$$
(\mathcal{O}Z)^I_{ij} = (\exp[2\pi i(m_i + m_j + m_a + m_b)/N] Z^{ab}_{ij}, \exp[2\pi i(m_i + m_j)/N] Z^\alpha_{ij}).
\tag{3.2.11}
$$

The fields $\tilde{Z}_{ij}^{ab}$, surviving the $\mathcal{O}$ truncation have therefore the same index structure of the auxiliary fields in eq. (3.2.8).

The second set of scalar fields is made of the fields $Z_{ij}^{p\varepsilon}$. Here the index $p\varepsilon$ labels a generator of the DISG, of nonzero lattice momentum p and polarization tensor ε associated with the vertex operator $V^{p\varepsilon}$. With an appropriate choice of ε, we can write the action of $\mathcal{O}$ on these vertices as

$$(\mathcal{O}V)^{p\varepsilon} = \exp[2\pi i k(\varepsilon,\delta)/N]V^{\lambda^{-1}p\varepsilon}. \tag{3.2.12}$$

Here $k(\varepsilon,\delta)$ is a function of ε, δ^A (and p) taking values in $\{0,..,N-1\}$. The invariant fields, surviving the $\mathcal{O}$ truncation are

$$\hat{Z}_{ij}^{p\varepsilon} = \frac{1}{N}\sum_{l=0}^{N-1}\exp[2\pi i l(m_i+m_j+k(\varepsilon,\delta))/N]Z_{ij}^{\lambda^l p\varepsilon}. \tag{3.2.13}$$

Notice that, after truncation, it is in general no longer possible to choose a basis of generators made of eigenvalues of the internal momenta p.

By using equations (3.2.11) and (3.2.12), we may find how, in $N=1$ and $N=2$ orbifolds, the DISG algebra $\mathbf{g}$ breaks down into representations of some sub-algebra $\mathbf{h}$. The action of (3.2.11) and (3.2.12) on massless states for the $E_8\times E_8$ heterotic string was given in ref. [112]. There it was found that the massless families and gauge groups of cases a through e were

- Nine 27 of E_6, corresponding to the fields $Z_{4j}^{(i,27)}$, $i,j=1,2,3$ for case a.
- Nine 27 of E_6, corresponding to the fields $Z_{4j}^{(i,27)}$, $i,j=1,2$ and $Z_{43}^{(3,27)}$ for case b.
- Three 27 of E_6, corresponding to the fields $Z_{4i}^{(i,27)}$, $i=1,2,3$ for case c.
- Four 27 of E_6, corresponding to the fields $Z_{4j}^{(i,27)}$, $i,j=1,2$ plus a 27: $Z_{43}^{(3,27)}$ and a $\overline{27}$: $Z_{43}^{(\overline{3},\overline{27})}$ for case d.
- Two 27 of E_6, corresponding to the fields $Z_{4i}^{(i,27)}$, $i=1,2$ plus a 27: $Z_{43}^{(3,27)}$ and a $\overline{27}$: $Z_{43}^{(\overline{3},\overline{27})}$ for case e.

The notation used above is the same as in our previous analysis: the E_8 gauge indices, between parenthesis, have been decomposed in $SU(3)$ and E_6 gauge indices.

One may recover the $N=1$ superpotential of all $N=1$ models by recalling that in $N=1$ the gravitino mass $m_{3/2}$ is related to the superpotential W by [61, 59]

$$m_{3/2} = \exp(K/2)W, \tag{3.2.14}$$

where K is the Kähler potential of the $N=1$ scalar manifold. In order to write the superpotential W containing all the fields that are relevant to low energy physics, i.e. those that may

become massless on some background, we introduce the following notation

$$\begin{aligned} y_i^p &= \exp(-K/6)\hat{Z}_{i4}^p, \quad y_i^A = \exp(-K/6)\frac{1}{N}\sum_{n=0}^{N-1}\mathcal{O}^n Z_{i4}^A, \\ & i=1,2,3, \quad p\eta p = 2. \end{aligned} \tag{3.2.15}$$

Thus the fields y are, up to a rescaling, those invariant under the transformations in (3.2.11), (3.2.12). Notice that here, as in eq. (3.2.13), p denotes an orbit of the action of $\mathbf{Z}_N$ on $\Gamma^{(6,22)}$.

In terms of the fields (3.2.15) the gravitino mass surviving the $N=1$ truncation reads (cfr. [112, 143, 144])

$$m_{3/2} = \exp(K/2)\epsilon^{ijk}[\sum_{p,q}\sum_{l=0}^{N-1} y_i^p y_j^q y_k^{p+\lambda^l q}\varepsilon(p,\lambda^l q) - \sum_{p,A} i p_A y_i^A y_j^p y_k^{-p}], \tag{3.2.16}$$

where ϵ^{ijk} is the totally antisymmetric tensor, $i,j,k=1,2,3$. The y^A are constrained fields, obeying the appropriate $N=1$ reduction of eq. (2.7.6) [112]. In order to find the superpotential in terms of physical fields one must first solve that constraint. Confronting eq. (3.2.16) with eq. (3.2.14) one easily gets the superpotential of a $\mathbf{Z}_N$ $N=1$ truncation. On the massless modes it coincides with that of ref. [112].

The study of the $N=1$ models is completed by specifying the form of the gauge coupling function, i.e. the function multiplying the square spin-one field strength term in the Lagrangian [61, 59]. This function f turns out to be model independent and equal to $f=\eta^{\alpha\beta}S$, where $\eta^{\alpha\beta}$ is the invariant metric restricted to the gauge fields surviving the $N=1$ truncation, and $S=\sqrt{G}\exp\phi + ia$. Here $\Re S$ is the universal duality-invariant dilaton field [283, 106, 112], and a is the axion.

We turn now to the study of truncations to $N=2$ orbifolds. The two geometrically distinct cases of $N=2$ orbifolds can be studied by the same methods used for the $N=1$ orbifolds. In $N=2$ supergravity the manifold of scalar fields reads $SU(1,1)/U(1)\times\mathcal{K}\times\mathcal{Q}$. The manifold $\mathcal{K}$ is Kählerian and $\mathcal{Q}$ is quaternionic [16, 79, 77]. The first $N=2$ case corresponds to the choice $m_1=m_2=N/2$, $m_3=m_4=0$, while the second one corresponds to $m_1\neq m_2$, $m_1+m_2=N$, $m_3=m_4=0$.

By repeating the same analysis done for the $N=1$ truncations one finds the following manifolds

$$\begin{aligned} \mathcal{K} &= \frac{O(2+n,m,\mathbf{R})}{O(2,\mathbf{R})\times O(n,m,\mathbf{R})}, \quad \mathcal{Q} = \frac{O(4+n',m',\mathbf{R})}{O(4,\mathbf{R})\times O(n',m',\mathbf{R})}, \quad m_1=m_2, \\ \mathcal{K} &= \frac{O(2+n,m,\mathbf{R})}{O(2,\mathbf{R})\times O(n,m,\mathbf{R})}, \quad \mathcal{Q} = \frac{SU(2+n',m',\mathbf{R})}{U(1)\times SU(2,\mathbf{R})\times O(n',m',\mathbf{R})}, \quad m_1\neq m_2, \\ & n=n'=m=m'=\infty. \end{aligned} \tag{3.2.17}$$

At this point we stop our by no means systematic review of the untwisted sector of $N = 1$ and $N = 2$ theories resulting from orbifold compactifications of $D = 4$, $N = 4$ strings. We recall that we confined our attention to $\mathbf{Z}_N$ symmetric orbifolds only. Many other types of orbifolds can be constructed. For example, one may study Abelian orbifolds generated by noncyclic groups like $\mathbf{Z}_2 \times \mathbf{Z}_2$ [10, 189], or non-Abelian orbifolds [86]. By repeating the procedure of this section one can work out what the $N = 1$ and $N = 2$ truncations look like in these cases.

By construction, each truncated model is invariant under the automorphism group of the corresponding $N = 1, 2$ DISG sub-algebra. The automorphism group contains the subgroup of $O(6, 22, \mathbf{Z})$ which stabilizes the orbifold twist group, that is the generalized duality group acting on the untwisted sector moduli space of the orbifold compactifications. These automorphism transformations are the residual discrete symmetries of the broken gauge group in the untwisted-sector effective action.

3.3 Special Geometry and Calabi-Yau Compactifications

Orbifolds correspond to special points in the space of all possible heterotic compactifications. A more general class of string vacua is given by the Calabi-Yau (CY) compactifications, giving rise to 4-D theories with $N = 1$ supersymmetry.

CY spaces arise when one reduces the ten-dimensional heterotic string to its point-field limit, namely, when one studies the 10-D effective action which describes physics at a scale $R \gg \alpha'^{1/2}$. This action, which is uniquely fixed by 10-D $N = 1$ supersymmetry, and by the knowledge of the gauge group, is then compactified to yield a 4-D theory. The resulting 4-D theory is a good approximation of the true 4-D LEEA, obtained by computing string amplitudes in 4-D, provided the compactification radii are much larger than $\alpha'^{1/2}$.

From the point of view of the underlying worldsheet conformal theory, these compactifications are (2,2) superconformal theories [129]. Orbifolds which are (2,2) superconformal correspond to particular (singular) points in the moduli space of a CY manifold [42]. Conversely, by giving nonzero expectation values to some twisted moduli, (2,2) orbifold singularities are blown up to yield smooth CY spaces. In general, the structure of the LEEA of CY compactifications is not exactly known, not in even the field-theoretical (large-radius) limit. Nevertheless, the Lagrangian corresponding to the flat directions (moduli) of the CY compactification obeys severe constraints. These constraints are not enough to fix the ("small") duality group, but they are sufficient to determine how the duality group acts on the moduli.

We shall follow ref. [40] in our review of properties of CY spaces. Let us recall that a Calabi-Yau manifold is a compact Kähler manifold of complex dimension three and holonomy $SU(3)$. A generic 3-D Kähler manifold has holonomy $U(3)$, instead. The vanishing of the $U(1)$ holonomy is equivalent to saying that a CY space admits a Ricci-flat metric [38]. The moduli

of a CY space $\mathcal{M}$ are given by those variations of the metric g_{mn} which preserve Ricci-flatness

$$R_{mn}(g+\delta g) = R_{mn}(g) = 0. \tag{3.3.1}$$

By writing these variations in complex coordinates μ, $\bar{\mu}$ etc., thanks to special properties of Kähler manifolds, one finds that $\delta g_{\mu\bar{\nu}}$ and $\delta g_{\mu\nu}$, $\delta g_{\bar{\mu}\bar{\nu}}$ *separately* obey eq. (3.3.1). Thus, the deformations of a given CY space are of two types: mixed (one holomorphic, one anti-holomorphic index), and pure (all holomorphic or all anti-holomorphic indices). To mixed variations one associates a real (1,1)-form

$$i\delta g_{\mu\bar{\nu}} dx^{\mu} \wedge dx^{\bar{\nu}}, \tag{3.3.2}$$

whereas to pure variations one associates a complex (2,1)-form by using the holomorphic (3,0) form Ω present in all CY manifolds [38, 42]:

$$\Omega^{\bar{\nu}}_{\kappa\lambda} \delta g_{\bar{\mu}\bar{\nu}} dx^{\kappa} \wedge dx^{\lambda} \wedge dx^{\bar{\mu}}. \tag{3.3.3}$$

These variations are holomorphic, thanks to eq. (3.3.1). Thus, the deformations of a CY manifold are elements of the Dolbeault cohomology groups $H^{(1,1)}$ and $H^{(2,1)}$ (see ref. [158] for a simple introduction to the subject).

The mixed variations are deformations of the Kähler class, in other words, they leave the complex structure invariant and change the metric. In particular, they change the *compactification radii*.

The pure variations, instead, modify the complex structure of the CY manifold (that is the split into holomorphic and anti-holomorphic coordinates).

Among the fields of the 10-D effective action of the heterotic string, there is a two form $B = b_{mn} dx^m \wedge dx^n$. This form and the Kähler form $J = g_{mn} dx^m \wedge dx^n$ can be written as a single complex form $B + iJ$. A variation of B gives rise to a massless mode in 4-D iff $\delta B \in H^{(1,1)}$ [42]. Thus, the most general metric for δg and δB writes as follows

$$ds^2 = \frac{1}{2V} \int_{\mathcal{M}} g^{1/2} d^6x g^{\kappa\bar{\mu}} g^{\lambda\bar{\nu}} [\delta g_{\kappa\lambda} \delta g_{\bar{\mu}\bar{\nu}} + (\delta g_{\kappa\bar{\nu}} \delta g_{\lambda\bar{\mu}} + \delta b_{\kappa\bar{\nu}} \delta b_{\bar{\mu}\lambda})]. \tag{3.3.4}$$

This equation has another important consequence: it says that the metrics of $(1,1)$ and $(2,1)$ moduli are factorized. So, at least locally, the moduli space of a CY space is $\mathcal{M}_{(1,1)} \times \mathcal{M}_{(2,1)}$. This property was proven in [253] using effective-theory arguments. In [87] it was shown that the factorization of moduli space is exact to all orders in α', being a property of any $(2,2)$ superconformal worldsheet field theory.

The factorization of moduli space is at the root of the particular kind of target space duality called "mirror symmetry" [159, 41]. In the present context, this symmetry transforms a CY manifold into another one, in which the role of $(1,1)$ and $(2,1)$ forms are interchanged. It also has another important consequence. Since $\mathcal{M}_{(2,1)}$ does not depend on $\mathcal{M}_{(1,1)}$, it can be evaluated

in the regime where all compactification radii $\alpha'^{-1/2}R \to \infty$. In this limit, the field-theoretical result is *exact* in α', and not merely a lowest order result. This statement is very strong: it says that the metric of the $(2,1)$ moduli does not receive sigma-model corrections.

The metric of $(1,1)$ moduli, however, does receive sigma-model corrections. In some exceptional case though, by using the mirror symmetry, and by finding a pair of mirror-conjugated manifolds, one obtains exact results for $\mathcal{M}_{(1,1)}$ as well [41, 159, 14].

We shall not discuss further these developments though, but instead restrict our attention to the (2,1)-moduli space and figure out its special properties.

The relation between (2,1) forms and deformations of the complex structure is given by [40]

$$\chi_{a\,\kappa\lambda\bar{\mu}} = -\frac{1}{2}\Omega^{\bar{\nu}}_{\kappa\lambda}\frac{\partial g_{\bar{\mu}\bar{\nu}}}{\partial z^a}, \quad \chi_a = \frac{1}{2}\chi_{a\,\kappa\lambda\bar{\mu}}dx^\kappa \wedge dx^\lambda \wedge dx^{\bar{\mu}}. \tag{3.3.5}$$

The z^a, $a = 1, ..., b_{21}$ are the parameters for the complex structure, whose number is equal to the dimension b_{21} of $H^{(2,1)}$. Equation (3.3.5) can be inverted to give $\delta g_{\mu\bar{\nu}}$ in terms of χ_a. By simple algebra one finds the definition of the metric of the (2,1)-moduli, $G_{a\bar{b}}$,

$$G_{a\bar{b}}\delta z^a \delta z^{\bar{b}} \equiv \frac{1}{2V}\int_{\mathcal{M}} g^{1/2}d^6x g^{\kappa\bar{\mu}}g^{\lambda\bar{\nu}}\delta g_{\kappa\lambda}\delta g_{\bar{\mu}\bar{\nu}} = -\frac{2i}{V\|\Omega\|^2}\delta z^a \delta z^{\bar{b}}\int_{\mathcal{M}}\chi_a \wedge \bar{\chi}_{\bar{b}}. \tag{3.3.6}$$

The (2,1) forms χ_a can be expressed in terms of the holomorphic three form as [40]

$$\frac{\partial\Omega}{\partial z^a} = \phi_a\Omega + \chi_a, \tag{3.3.7}$$

where the ϕ_a may depend on z^a, but not on the coordinates of the CY space.

Equation (3.3.7) implies that the metric of (2,1) forms is Kähler, i.e.

$$G_{a\bar{b}} = -\frac{\partial}{\partial z^a}\frac{\partial}{\partial z^{\bar{b}}}\log\left(i\int_{\mathcal{M}}\Omega\wedge\bar{\Omega}\right). \tag{3.3.8}$$

Indeed, this metric enjoys other properties. To study them, one picks up a (real) canonical homology basis (A^i, B_j), $i, j = 0, ..., b_{21}$ of the *integral* homology group $H_3(\mathcal{M}, \mathbf{Z})$. The dual cohomology basis (α_i, β^j) obeys

$$\int_{A^i}\alpha_j = \int_{\mathcal{M}}\alpha_j\wedge\beta^i = \delta^i_j, \quad \int_{B_i}\beta^j = \int_{\mathcal{M}}\beta^j\wedge\alpha_i = -\delta^j_i. \tag{3.3.9}$$

Notice that this equation defines a symplectic metric on $H_3(\mathcal{M}, \mathbf{Z})$, given by the wedge product of two three-forms. In terms of this cohomology basis the holomorphic three-form Ω reads

$$\Omega = x^i\alpha_i - \mathcal{G}_j\beta^j. \tag{3.3.10}$$

The dimension of the (2,1)-moduli space is b_{21}, therefore only b_{21} of the $2b_{21}+2$ coefficients in eq. (3.3.10) are independent. One can choose $x^i = \lambda z^i$, $i = 1, .., b_{21}$, $x^0 = \lambda$, and express $\mathcal{G}_i$ in

function of the x^i. The parameter λ is an irrelevant scale factor thus, obviously, $\mathcal{G}_i(\lambda x) = \lambda \mathcal{G}_i(x)$. Equation (3.3.7) implies

$$\int_{\mathcal{M}} \Omega \wedge \frac{\partial \Omega}{\partial x^i} = 0. \tag{3.3.11}$$

This equation says that $\mathcal{G}_i$ is the derivative of a homogeneous function of degree two: $\mathcal{G}_i = \partial \mathcal{G}/\partial x^i$, and $\mathcal{G}(\lambda x) = \lambda^2 \mathcal{G}(x)$. The Kähler potential of the (2,1)-moduli space is therefore expressible in terms of a holomorphic function:

$$\int_{\mathcal{M}} \Omega \wedge \bar{\Omega} = \bar{x}^i \frac{\partial \mathcal{G}}{\partial x^i} - x^i \frac{\partial \mathcal{G}}{\partial \bar{x}^i}. \tag{3.3.12}$$

This particular form of the Kähler potential is the one dictated by $N = 2$ supersymmetry in 4-D, for the scalars of the vector multiplet [79, 77].

The link between $N = 2$ supersymmetry and the geometry of (2,1) moduli is not an accident. As shown in ref. [253], the moduli of a (2,2) heterotic compactification, with $N = 1$ target space supersymmetry, are the same as for a type-II superstring compactification, with $N = 2$ supersymmetry. The map between heterotic compactifications and type-II superstring compactifications puts stronger constraints on the (2,1)-moduli Lagrangian than $N = 1$ supersymmetry alone. These constraints, which follow from $N = 2$ supersymmetry, have been studied in refs. [48, 268, 102]. Moreover, one can show that even the (1,1)-moduli Lagrangian can be obtained from a type II compactification. Therefore, even for (1,1) moduli, the form of the Kähler potential is of special type:

$$K = -\log i \left(\bar{y}^m \frac{\partial \mathcal{F}}{\partial y^m} - y^m \frac{\partial \mathcal{F}}{\partial \bar{y}^m} \right), \quad m = 0, ..., b_{11}. \tag{3.3.13}$$

Here the prepotential $\mathcal{F}$ is corrected by $\mathcal{O}(\alpha')$ terms, and the link between duality and CY geometry is less transparent than for (2,1) moduli. For this reason, we prefer to return to this latter case, and examine it more closely.

We spent some time on the details of the construction of metrics for the (2,1)-moduli space. These details allow us to find a general *exact* result about target space duality in CY compactifications. First of all, let us notice that an infinitesimal change of the coordinates z^a does not change the cohomology basis in eq. (3.3.9), since the homology group is a topological invariant of the CY space, and its coefficients are integers. On the other hand, under a large diffeomorphism $z^a \to z'^a(z)$, the homology basis could undergo a symplectic transformation

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(b_{21} + 1, \mathbf{Z}). \tag{3.3.14}$$

The coefficients A, B, C, D are integer valued, since the homology is integral, and they define a symplectic matrix, due to eq. (3.3.9). Equation (3.3.14) leads to a transformation rule for the periods of the three-form Ω given, in our notations, by the vector $(\partial_i \mathcal{G}, x^i)$

$$\begin{pmatrix} \frac{\partial \mathcal{G}}{\partial x^i} \\ x^i \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{G}}{\partial x^i} \\ x^i \end{pmatrix}. \tag{3.3.15}$$

This is a duality: two different backgrounds, related by transformation (3.3.15), yield the same physics, since they correspond to a reparametrization of the (2,1)-moduli space. The form of the transformation on the z^a coordinates is highly non-trivial, nevertheless, eq. (3.3.15) states that it acts as a linear symplectic transformation on $\partial_i \mathcal{G}$, x^i.

The previous result is totally general, valid for *any* CY compactification, and it is exact in α', since it concerns (2,1) moduli. Given a CY manifold $\mathcal{M}$, a similar result would hold for the (1,1) moduli, if a mirror manifold $\tilde{\mathcal{M}}$ could be found.

Clearly, eq. (3.3.15) is not sufficient to fix the duality group Γ of a CY space: it only says that Γ is a subgroup of $Sp(b_{21}+1, \mathbf{Z})$. Equation (3.3.15) also defines the action of Γ on the periods x^i, $\partial_i \mathcal{G}$.

Our analysis of the CY moduli space has linked the action of the duality group to algebraic geometric properties of the CY space. The explicit form of the duality group Γ can be found in some cases, again by use of algebraic geometry. In ref. [41], Γ was found in the simple case of CY manifolds described by a one complex-parameter family of quintic hypersurfaces in $\mathbf{CP}_4$. Even in that simple case, it turns out that Γ is not $SL(2, \mathbf{Z})$, as previously conjectured in the literature.

A further development, that we shall only mention here, is given by the relation between Γ and the monodromy of Picard-Fuchs equations for the periods of CY spaces [41, 213, 37, 53, 70]. These are the differential equations giving the periods x^i, $\partial_i \mathcal{G}$ as functions of the complex-structure moduli z^a. These equations are singular at some point(s). By transporting z^a about that singular point, the solutions of the differential equations transform into each other. The group describing this transformation is called the monodromy group Γ_M. Almost all known examples of CY manifolds are representable as algebraic hypersurfaces in (some) $\mathbf{CP}_n$ by the equation

$$W_i(y, z) = 0, \quad y \in \mathbf{CP}_n, \quad z \in \mathcal{M}_{(2,1)}. \tag{3.3.16}$$

Here W_i are quasi-homogeneous polynomials in y, z. Associated with eq. (3.3.16) there exists an invariance group Γ_W, whose elements are quasi-homogenous transformations of y, z leaving the equation invariant. In this case, the authors of ref. [213] proposed a general formula for the duality group, namely $\Gamma/\Gamma_W = \Gamma_M$. In other words, they conjectured that the duality group is a semi-direct product of the invariance group of equation (3.3.16), and the monodromy group of the periods $\Gamma = \Gamma_W \otimes_S \Gamma_M$. This conjecture holds true in the case of a single modulus [41, 213, 70].

The proposed formula for the duality group was shown to hold true for K_3 compactifications, relevant for $N = 2$ heterotic backgrounds, in ref. [150]. Moreover, as was studied in [150], the invariance group Γ_W is the symmetry group of the corresponding $N = 2$ Landau-Ginzburg (LG) model. This LG model and its corresponding CY manifold are "different phases" of an underlying $N = 2$ superconformal field theory [223, 276, 161, 214, 287].

Finally, we stress again that informations about $\mathcal{M}_{(1,1)}$ cannot be obtained simply by algebraic geometry. Only in the case when a mirror to the CY manifold $\mathcal{M}$ exists, one can extract exact results about (1,1) moduli.

3.4 Duality Invariant $N = 1$ Actions: Superpotential, Threshold Effects and Integration of Massive Modes

Any LEEA describing a 4-D compactification must be invariant under the appropriate duality group. In particular, even the "small" duality group, transforming massless modes into themselves, dictates useful restrictions of the form of the LEEA.

In section 3.1, we have examined the form of duality transformations on twisted and untwisted fields in some orbifold compactifications. For the simple case of the $\mathbf{Z}_3$ orbifold we found that all massless vertices transform in the following way, up to a moduli-independent phase

$$\phi_s \to \left(\frac{c_i\bar{\tau}_i + d_i}{c_i\tau + d_i}\right)^{-n_i^s/2} \phi_s. \tag{3.4.1}$$

Here, s labels the massless scalar fields and the charges n_i^s are given in eqs. (3.1.26, 3.1.30). The fields ϕ_s are those linear superposition of the vertices of section 3.1 that diagonalise the unitary matrices S^1_{ab}, S^2_{ab} of eq. (3.1.30). The fields A_s, defined by

$$\phi_s = i\prod_{i=1}^{3}(\tau_i - \bar{\tau}_i)^{n_i^s/2} A_s, \tag{3.4.2}$$

transform under $SL(2,\mathbf{Z})^3$ as modular forms of weight (n_1^s, n_2^s, n_3^s)

$$A_s \to \prod_{i=1}^{3}(c_i\tau_i + d_i)^{n_i^s} A_s. \tag{3.4.3}$$

An $N = 1$ Lagrangian is determined, as we recalled in section 3.2, by the Kähler potential K, giving rise to the kinetic term of the scalar-multiplet fields, the superpotential W, and the field-dependent gauge coupling constant f^{AB}. The superpotential is a holomorphic function of the scalar fields, and the indices A,B range on the generators of the gauge group. The Kähler potential for a LEEA of a (2,2) compactification of the heterotic string can be expanded, around a given value of the moduli, in powers of the fields A_s. This potential can be evaluated at tree-level by computing string scattering amplitudes on the sphere [87]. The expansion to quadratic terms in the charged fields A_s reads, for the case of our $\mathbf{Z}_3$ orbifold [87, 110, 177, 176]

$$K = -\sum_{i=1}^{3}\log(2\Im\tau_i) + \prod(2\Im\tau_i)^{n_i^s}|A_s|^2. \tag{3.4.4}$$

This result coincides with the Kähler potential one finds by dimensional reduction of 10-D supergravity coupled to $E_8 \times E_8$ gauge matter [108].

Notice that the Kähler potential in eq. (3.4.4) transforms as follows under $SL(2,\mathbf{Z})^3$ duality

$$K \to K + \Lambda + \bar{\Lambda}, \quad \Lambda = \sum_{i=1}^{3} (c_i \tau_i + d_i). \tag{3.4.5}$$

The kinetic term for the gauge fields is diagonal within each simple factor G_a of the gauge group $G = \prod_a G_a$. At tree-level one finds [131]

$$f_a^{AB} = k_a \delta^{AB} S. \tag{3.4.6}$$

The integer k_a is the level of the affine Lie algebra associated with G_a, and S is the axion+dilaton field. In the case of the $\mathbf{Z}_3$ orbifold $G = SU_3 \times E_6 \times E_8$, and all levels are equal to one. An $N = 1$ Lagrangian depends on the superpotential W and Kähler potential K only through the combination

$$\mathcal{G} = K + \log |W|^2. \tag{3.4.7}$$

Thus, the LEEA of our orbifold compactification is duality invariant if

$$W(A_s, \tau_i) \to \lambda W(A_s, \tau_i) \prod_{i=1}^{3} (c_i \tau_i + d_i)^{-1}, \tag{3.4.8}$$

with λ an arbitrary *moduli independent* phase. This property has been verified at tree-level in $\mathbf{Z}_N$ orbifold compactifications in refs. [208, 57, 212]. One may also use the transformation properties of the fields A_s, and the form of the Kähler potential, to predict the form of the superpotential [110]. More importantly, since duality is an exact *all-loop* symmetry, radiative corrections should be duality invariant [109, 110].

If one further assumes that duality, besides being an all-order symmetry, is also preserved non-perturbatively, one may deduce useful constraints on supersymmetry breaking mechanisms [114, 64, 111, 239, 30, 120].

Let us find how duality constrains the form of the superpotential. We shall at first give general arguments, based on duality invariance alone. To simplify the discussion as much as possible, let us keep only one modulus τ in the LEEA, and consider the corresponding $SL(2,\mathbf{Z})$ duality. In this case, there are two possibilities of implementing $SL(2,\mathbf{Z})$ [109]. The first one is that the Kähler potential K and the superpotential W be separately $SL(2,\mathbf{Z})$-invariant. A possible solution to this constraint is to modify K (and W) so as to recover the tree-level form only in the large $\Im\tau$ limit. If the tree-level Kähler potential is chosen, in analogy with (3.4.4) to be

$$K = -n \log(2\Im\tau), \tag{3.4.9}$$

then a possible modular invariant completion is [146, 109]

$$K = -n \log \left[\sum_{m,n \in \mathbf{Z}} \exp\left(-\frac{\pi}{\Im\tau} |m + n\tau| \right) \right]. \tag{3.4.10}$$

The second possibility is that both K and W transform under duality, and only the combination $K + \log|W|^2$ is invariant. When K is given by eq. (3.4.9), duality invariance demands that $W(\tau) \to (c\tau+d)^{-n}W(\tau)$. This means that the superpotential must transform as a modular form of weight $-n$. If we want W to be analytic at any finite value of the modulus τ, inside the fundamental domain of the upper half-plane $|\tau| \geq 1$, $-1/2 \leq \Re\tau \leq 1/2$, we find [109]

$$W(\tau) = \text{const}\,\eta(\tau)^{-2n} = e^{n\pi\Im\tau/6}[1 + 2ne^{-2\pi n\Im\tau} + 2n(n+1)e^{-4\pi n\Im\tau} + ...]. \tag{3.4.11}$$

This superpotential has the form expected from worldsheet instantons, and integration of massive modes. In ref. [144] it was suggested that modular forms as in eq. (3.4.11) may originate from the integration of massive terms in $N = 1$ LEEAs given by eq. (3.2.16). Indeed, those LEEAs contain an infinite number of (untwisted) massive fields, say M_U. Some of these fields may appear linearly in the superpotential: $W = M_U L_T L_T$. By L_T we denote the massless (light) twisted scalars. Upon integration of the massive fields, the original trilinear superpotential of the LEEA is modified. This modification must still be invariant under the "small" duality group $SL(2,\mathbf{Z})$. The resulting superpotential, which depends only on the massless fields, must therefore be a modular form of appropriate weight, as in eq. (3.4.11).

Our discussion has been so far quite general. No input on the form of the LEEA, besides duality and supersymmetry, has been used. In particular, no physical mechanism has been proposed for the appearance of automorphic modular forms. Now, we would like to study in some more detail a mechanism whereby non-trivial modular functions modify tree-level physics.

The most dramatic modification to tree-level results affects the gauge couplings. The tree-level result (3.4.6), indeed, states that these couplings depend on the dilaton field S only. In particular one has the tree-level identification $1/g_a^2 = k_a\Re S$ for the gauge coupling constant g_a. At one loop this equation is modified (as it happens in field theory too). In fact, one must specify the scale μ at which the coupling constant is defined, and take into account the renormalization-group running of g_a as well as threshold effects due to the integration of massive modes [183]

$$\frac{1}{g_a^2(\mu)} = k_a\Re S + \frac{b_a}{16\pi^2}\log\frac{M_{string}}{\mu^2} + \Delta_a. \tag{3.4.12}$$

Here b_a is the one-loop coefficient of the beta function, equal to $-3C(G_a) + h_j\sum_j T(R_j)$. $C(G_a)$ is the quadratic Casimir of the adjoint representation of the gauge group, $T(R_j)$ is the Casimir of the R_j representation, $h_j = 1$ for spin-1/2 fermions, and $h_j = 1/4$ for scalars. The sum runs over all massless fields. The coefficient Δ_a represents the threshold effects at the string unification scale M_{string}. Its form has been found explicitly in ref. [88] for the case of (2,2) orbifolds:

$$\Delta_a(\tau) = \frac{1}{16\pi^2}b'_a\log 2\Im\tau|\eta(\tau)|^4, \quad b'_a = -b_a - 2\sum_j h_j T(R_j)(1+n_j). \tag{3.4.13}$$

The n_j denote the modular weights of the matter fields. Equation (3.4.13) is invariant under $SL(2,\mathbf{Z})$ duality. This happens because $\Im\tau$ transforms under duality so as to cancel the

transformation of $|\eta(\tau)|^4$ functions.

It is interesting to notice that the eta-function contribution in Δ_a is the real part of an holomorphic function, and comes from the loop integration over massive modes [88]. The $\Im\tau$ contribution instead, arising from massless loops, cannot be written as the real part of an analytic function. This seems to contradict supersymmetry, which requires the gauge-coupling function f^{AB} to be holomorphic (chiral). The conundrum is solved by noticing that a non-holomorphic coupling is compatible with supersymmetry, if one allows non-local terms in the effective Lagrangian [73]. Namely, one finds that it is possible to supersymmetrize the gauge kinetic term

$$\int d^4x h(z,\bar{z}) F^a_{\mu\nu} F^{a\,\mu\nu}, \quad z = \text{scalar field}, \tag{3.4.14}$$

even when

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} h(z,\bar{z}) \neq 0. \tag{3.4.15}$$

The superfield expression for the gauge kinetic term is [73]

$$\frac{1}{4}\int d^2\theta d^4x \mathcal{P} h(z,\bar{z}) W^{a\alpha} W^a_\alpha + \text{h.c.}, \qquad \mathcal{P} = -\frac{1}{16}\Box^{-1}\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}\mathcal{D}. \tag{3.4.16}$$

Here z is a chiral superfield, W^a_α is the supersymmetric gauge field-strength, and $\mathcal{P}$ is a chiral projector. This expression reduces to the standard one when $h(z,\bar{z}) = h(z)$. A non-local term may arise from the integration of massless modes, but not of massive ones. We have just seen that the only nonlocal term in (3.4.15) is $\log \Im\tau$, which indeed arises from massless loops [88].

The threshold correction $\Delta_a(\tau,\bar{\tau})$ given by eq. (3.4.13) gives rise to a significant modification of the gaugino-condensate mechanism of supersymmetry breaking [85]. The main feature of this mechanism is that in the presence of a hidden sector, and if some gauge interactions become strong, a non-vanishing condensate of strongly interacting fields may form. This means that if E_8' is unbroken, for instance, gauginos may condense [105]. This condensate modifies the effective superpotential below the condensation scale Λ_c, and may lead to supersymmetry breaking at zero cosmological constant. The gaugino-condensate dependence on the coupling constant is dictated by renormalization-group invariance

$$\langle\bar{\lambda}\lambda\rangle = \Lambda_c^3 \exp\left(24\pi^2 b_H \frac{1}{g_H^2}\right). \tag{3.4.17}$$

Here the only relevant gauge coupling is the one of the hidden strongly interacting gauge group (e.g. E_8'), called g_H. By substitution of eqs. (3.4.12, 3.4.13) into formula (3.4.17), one finds a VEV for $\langle\bar{\lambda}\lambda\rangle$ which depends on the modulus τ. Since the threshold correction function Δ_a is modular invariant, so is $\langle\bar{\lambda}\lambda\rangle$. Assuming again that no fields in a chiral representation of the hidden group G_H is present, one finds $b_H = -b'_H$. In this case the gaugino condensate (3.4.17) produces a very simple modification of the superpotential [114, 64]

$$W(S,\tau) = \frac{e^{24\pi^2 S/b_H}}{\eta^6(\tau)}. \tag{3.4.18}$$

This superpotential has modular weight -3. Thus, together with the Kähler potential $-3\log(2\Im\tau)$, it gives rise to a $SL(2,\mathbf{Z})$ invariant LEEA. Equation (3.4.18) can be generalized so as to take into account a possible non-perturbative modification of the S-dependent part of the superpotential

$$W(S,\tau)=\frac{\Omega(S)}{\eta(\tau)}. \tag{3.4.19}$$

For instance, in ref. [85] it was assumed $\Omega(S)=c+\exp(24\pi^2 S/b_h)$. Notice that modular invariance does not give any constraint on $\Omega(S)$.[23] The superpotential (3.4.19), together with the standard Kähler potentials for τ and S, gives rise to the scalar potential [114]

$$V=\frac{1}{16\Re S\Im\tau|\eta(\tau)|^{12}}\left[|2\Re S\Omega_S-\Omega|^2+3|\Omega|^2\left(\frac{(\Im\tau)^2}{\pi^2}|\hat{G}_2|^2-1\right)\right]. \tag{3.4.20}$$

Here $\Omega_S=\partial_S\Omega$, and $\hat{G}_2$ is the weight-2 (non-holomorphic) Eisenstein function (see for instance [258]). The functional dependence of (3.4.20) on τ implies that the potential diverges both at $\Im\tau\to\infty$ and $\Im\tau\to 0$. By recalling that $\Im\tau$ is the compactification radius of the internal manifold (orbifold) one draws an important conclusion: gaugino condensation *and duality* force string theory to compactify on spaces of size $\mathcal{O}(\alpha'^{1/2})$. The potential in eq. (3.4.20) has been generalized to the case when G_H-chiral matter is present in ref. [216].

In ref. [177] it was also noticed that potential (3.4.20) gives rise to *real* soft supersymmetry-breaking terms, owing to the fact that minima lie at $\Re\tau$=integer. This fact guarantees the smallness of CP-violating terms in the low-energy effective action.

Duality invariant gaugino condensation has also been studied in [111, 30, 239]. In particular, in ref. [111], gaugino condensation was analysed by introducing an effective Lagrangian including a composite chiral superfield $U=W^{a\alpha}W^a_\alpha$, which describes the degrees of freedom arising from the condensate. The resulting Lagrangian is the analog of the pion Lagrangian in QCD, with U playing the role of the composite pion field. The scalar potential found in ref. [111] and the resulting supersymmetry breaking substantially agree with [114, 64] and [30, 239]. The analysis of non-perturbative supersymmetry breaking in string theory has further been studied, for instance, in refs. [30, 217, 46, 181, 73].

Duality-invariant functions, as well as functions transforming with a definite weight under duality (automorphic dual functions) have entered our discussion of threshold effects and gaugino condensation in several places. For instance, we saw that threshold effects in the one-loop gauge coupling constant are represented by eq. (3.4.13) as $\log\|\eta^{-3}\|^2$, with the norm $\|..\|^2$ given by $e^K|..|^2$. Threshold effect, therefore, can be thought as given by a non-holomorphic norm of the modular form η. The construction of this norm, and of generalized modular forms, has been

[23] In ref. [115] $\Omega(S)$ was determined by conjecturing a new duality invariance, acting on the field S. The existence of this new symmetry has not yet been proven within the present formulation of string theory. Calculations supporting this conjecture were performed in ref. [259, 257, 252].

extended to the case of (2,2) compactifications in ref. [107]. There it was observed that, even though the duality group of (2,2) compactifications is unknown in general, yet its action on the moduli is known. Indeed, the moduli of $(2,2)$ compactifications parametrize a special Kähler manifold, due to the correspondence between (2,2) $N=1$ heterotic compactifications, and type II, $N=2$ compactifications. By calling M the dimension of the moduli space, and y^m its homogeneous coordinates, the Kähler potential writes as in the CY case

$$K = -\log i\left(\bar{y}^m \frac{\partial \mathcal{F}}{\partial y^m} - y^m \frac{\partial \mathcal{F}}{\partial \bar{y}^m}\right), \quad m = 0, ..., M. \tag{3.4.21}$$

In order to respect the symplectic structure of (3.4.21), duality transformations must act on y^m,$\partial_m \mathcal{F}$ as $Sp(M+1, \mathbf{R})$ transformations. By analogy with the case of (2,1) moduli in CY spaces, we may suppose that the duality group $\Gamma \subset Sp(M+1, \mathbf{Z})$. Therefore, one may write down a natural ansatz for a Γ-invariant function $\mathcal{G}$ as

$$\mathcal{G} = \sum_{m_i, n^i} \log i \frac{|m_i y^i + n^i \mathcal{F}_i|^2}{y^i \bar{\mathcal{F}}_i - \bar{y}^i \mathcal{F}_i}, \quad \mathcal{F}_i \equiv \partial_i \mathcal{F}. \tag{3.4.22}$$

Here the sum ranges over all integers in an orbit of the group Γ. Notice that by writing $\exp \mathcal{G} = \exp(K)|\Delta(y)|^2$, one would get, formally

$$\Delta(y) = \prod_{m_i, n^i} (m_i y^i + n^i \mathcal{F}_i). \tag{3.4.23}$$

Here too, the sum ranges on an orbit of Γ. Equation (3.4.23) needs a regularization to be made rigorous. If this regularization exists (as in the case of $\mathbf{Z}_3$ orbifolds [103]) then equation (3.4.23) defines an automorphic function of the duality group Γ. In these formulae, $\mathcal{G}$ has the same structure of the threshold-effect function (3.4.13), whereas Δ is the analog of the Dedekind eta function $\eta(\tau)$.

4 Duality in Curved Backgrounds with Abelian Symmetries

In this section, we discuss the discrete symmetry group acting on the space of D-dimensional curved backgrounds that are independent of d coordinates, and some of its applications. For such backgrounds systematic knowledge is available.

We begin, in section 4.1, with a target space geometry with a compact Abelian symmetry. A duality transformation is then performed [33, 34, 35] by gauging this symmetry and adding a Lagrange multiplier term that constrains the gauge field to be pure gauge. We show that such a duality transformation acts on the (curved) background matrix as a *factorized duality* (2.4.29).

In section 4.2, we show [246] that this factorized duality is a target space duality, namely, it relates (different) backgrounds that correspond to the same CFT. Moreover, we find a duality (sub-)group acting on the space of D-dimensional curved backgrounds with d commuting, compact Abelian symmetries [147]. In section 4.3 we discuss the action of the continuous group, called $\mathcal{G}$ in section 2, generating a local neighborhood of the moduli space. In the case considered here, of backgrounds with $U(1)^d$ symmetries, $\mathcal{G} = O(d,d,\mathbf{R})$. Based on analogies with the flat case, we conjecture, in section 4.4, how these results extend to the heterotic string.

In sections 4.5-4.7 we present some applications. An explicit string background [20, 92, 222, 245] turns out to have an interpretation of a 2-d black hole [285]. It is instructive to study stringy features of propagation in this background. In section 4.5 we discuss the duality of the two-dimensional black hole [137, 83, 23]. In section 4.6, we study two examples: duality between compact black strings and charged black holes in the bosonic string, and duality between neutral and charged black holes in the heterotic string [147, 173]. In section 4.7, we discuss cosmological backgrounds with compact Abelian symmetries [202, 235, 142].

In section 4.8, we interpret duality in curved backgrounds as a spontaneously broken gauge symmetry of string theory [139]; this is in analogy with the discussion of the flat case (see section 2.6). Finally, in section 4.9, we discuss duality and topology change in string theory [139].

4.1 Factorized Duality in Non-Linear σ-Models

In this section we follow ref. [246]. Let us consider the σ-model action

$$S = \frac{1}{2\pi}\int d^2z\,(G_{\mu\nu}(x) + B_{\mu\nu}(x))\partial x^\mu \bar{\partial} x^\nu, \tag{4.1.1}$$

where G is the metric on some manifold and the antisymmetric background B is the potential for the torsion three-form $H = \frac{3}{2}dB$. The complex worldsheet coordinates and derivatives are

$$z \equiv \frac{1}{\sqrt{2}}(\tau + i\sigma), \quad \partial \equiv \frac{1}{\sqrt{2}}(\partial_\tau - i\partial_\sigma), \tag{4.1.2}$$

and therefore, $d^2z = d\sigma d\tau$, and $\partial x \bar{\partial} x = \frac{1}{2}((\partial_\tau x)^2 + (\partial_\sigma x)^2)$.

Suppose that the action (4.1.1) is invariant under the isometry

$$\delta x^\mu = \epsilon \mathbf{k}^\mu. \tag{4.1.3}$$

This happens when the vector $\mathbf{k}^\mu$ satisfies the Killing equation $\mathcal{L}_\mathbf{k} G_{\mu\nu} = \mathbf{k}_{\mu;\nu} + \mathbf{k}_{\nu;\mu} = 0$ and, in addition, $\mathcal{L}_\mathbf{k} H = 0$, where $\mathcal{L}_\mathbf{k}$ is the space-time Lie derivative. The latter implies that locally

$$\mathcal{L}_\mathbf{k} B = d\omega \tag{4.1.4}$$

for some one form ω. Here d is the space-time exterior derivative. For σ-models that have, in addition, a dilaton term

$$S_{dilaton} = -\frac{1}{8\pi}\int d^2z \phi(x) R^{(2)}, \tag{4.1.5}$$

the dilaton field ϕ must satisfy $\mathbf{k}^\mu \phi_{,\mu} = 0$.

With these conditions it is possible to choose coordinates $\{x^\mu\} = \{x^0, x^a\}$ such that the isometry acts by translation of $x^0 \equiv \theta$, and all the background fields, G, B and ϕ, are independent of θ. (Here one has to use the fact that B is defined only up to (space-time) gauge transformations, $B \to B + d\lambda$, for some one-form λ; the one-form ω defined in (4.1.4) transforms under this gauge symmetry as $\omega \to \omega + \mathcal{L}_\mathbf{k}\lambda$.) In the coordinates $\{\theta, x^a\}$, the action (4.1.1, 4.1.5) takes the form

$$S[x^a, \theta] = \frac{1}{2\pi}\int d^2z \Big[G_{00}(x^c)\partial\theta\bar{\partial}\theta + (G_{0a}(x^c) + B_{0a}(x^c))\partial\theta\bar{\partial}x^a + (G_{a0}(x^c) + B_{a0}(x^c))\partial x^a\bar{\partial}\theta + (G_{ab}(x^c) + B_{ab}(x^c))\partial x^a\bar{\partial}x^b - \frac{1}{4}\int d^2z\phi(x^c)R^{(2)}\Big]. \tag{4.1.6}$$

To get the dual theory, we gauge the Abelian symmetry by minimal coupling, $\partial\theta \to \partial\theta + A$, $\bar{\partial}\theta \to \bar{\partial}\theta + \bar{A}$, and add a Lagrange multiplier term $\tilde{\theta}F$, where $F = \partial\bar{A} - \bar{\partial}A$ is an Abelian field strength. Choosing a gauge $\theta = 0$, one finds the first-order action $S[x^a, A, \tilde{\theta}]$ [261, 118, 34]

$$S[x^a, A, \tilde{\theta}] = \frac{1}{2\pi}\int d^2z \Big[G_{00}A\bar{A} + (G_{0a} + B_{0a})A\bar{\partial}x^a + (G_{a0} + B_{a0})\partial x^a\bar{A} + (G_{ab} + B_{ab})\partial x^a\bar{\partial}x^b - \frac{1}{4}\int d^2z\phi R^{(2)} + \tilde{\theta}F\Big]. \tag{4.1.7}$$

The integration of the Lagrange multiplier field $\tilde{\theta}$ gives $F = 0$, implying that on a topologically trivial worldsheet the gauge fields are pure gauge, $A = \partial\theta$, $\bar{A} = \bar{\partial}\theta$. Thus, one recovers the

original model (4.1.1) (modulo some important global issues that will be discussed below). Integrating out the gauge fields $A, \bar{A}$ one finds the *dual* model; this is a *new* theory with action S' [34, 35]

$$S' = \frac{1}{2\pi}\int d^2z \left[(G'_{\mu\nu}(x^a) + B'_{\mu\nu}(x^a))\partial y^\mu \bar{\partial} y^\nu - \frac{1}{4}\int d^2z \phi'(x^a) R^{(2)}\right], \tag{4.1.8}$$

where $\{y^\mu\} = \{\tilde{\theta}, x^a\}$, and the dual background is related to the original one by the action of factorized duality in the x^0 direction (exactly as in the flat case in section 2.4 (2.4.29)). Explicitly:

$$\begin{aligned} G'_{00} &= \frac{1}{G_{00}}, \qquad G'_{0a} = \frac{B_{0a}}{G_{00}}, \qquad G'_{ab} = G_{ab} - \frac{G_{a0}G_{0b} + B_{a0}B_{0b}}{G_{00}} \\ B'_{0a} &= \frac{G_{0a}}{G_{00}}, \qquad B'_{ab} = B_{ab} - \frac{G_{a0}B_{0b} + B_{a0}G_{0b}}{G_{00}}. \end{aligned} \tag{4.1.9}$$

The dilaton field receives corrections from the Jacobian that comes from integrating out the gauge fields. At one loop, this leads to a shift in the dilaton [33, 34, 35]:

$$\phi' = \phi + \log G_{00}. \tag{4.1.10}$$

With this shift, if the original theory is conformal invariant, at least to one-loop order, the dual theory is conformal invariant as well.

The dual theory is independent of the coordinate $\tilde{\theta}$, and therefore, it also has an Abelian isometry. However, in general, the geometry of the σ-model is changed by the duality transformation.

Are the dual theories equivalent as CFTs? To answer this question in the affirmative, one ought to show that the two theories are equivalent on worldsheets of any genera. For this reason one has to consider global issues of the procedure [246, 148, 4]. Let us assume that the isometry corresponds to a *compact* $U(1)$ group, so that the coordinate θ is periodic with periodicity 2π. To recover the original model after integrating out $\tilde{\theta}$, namely, to get the correct periodicity for θ, the holonomies $h = \oint A$ along any non-trivial homology cycle of the worldsheet should be restricted to integer numbers. To restrict h one should add $\sum_n \delta(h = n)$ to the path integral. For example, for a worldsheet with the topology of a torus one may write

$$\sum_n \delta(h = n) = \sum_{n_a, n_b} e^{i(n_a \oint_a A + n_b \oint_b A)}. \tag{4.1.11}$$

In other words, the term $n_a \oint_a A + n_b \oint_b A$ is added to the action $S[x, A, \tilde{\theta}]$, and the integration measure of the path integral contains a sum over n_a, n_b. Integrating out $\tilde{\theta}$ gives the original model (4.1.1). Now, to integrate out A (by first doing partial integration on $\tilde{\theta}F$ to get $\partial\tilde{\theta}\bar{A} - \bar{\partial}\tilde{\theta}A$) we must interpret n_a, n_b as the winding modes of $\tilde{\theta}$ around the a- and b-cycles on the torus. This fixes the periodicity of $\tilde{\theta}$ to be 2π. With this periodicity the action (4.1.8) indeed is equivalent to the original model without the gauge field.

4.2 $O(d,d,\mathbf{Z})$ in Curved Backgrounds Independent of d Coordinates

We will now show that the factorized duality (4.1.9, 4.1.10) in the θ direction is a symmetry of the CFT. This symmetry can be generalized to all the elements of $O(d,d,\mathbf{Z})$ if the curved background possesses d Abelian symmetries [147].

To show this we first construct, in subsection 4.2.1, general (conformal) curved backgrounds in D dimensions, that are independent of d coordinates, as Abelian quotients [154, 22, 128, 164, 3, 186, 250, 187] of a "parent" CFT in a $(D+d)$-dimensional background. Then the proof will be done [246, 147] relating different backgrounds in D dimensions, by gauging in different ways – *axial* or *vector* – the d Abelian symmetries of the parent σ-model, and combining the different gaugings with some manifest symmetries of the resulting D-dimensional quotients.

In subsection 4.2.2, we use the construction of subsection 4.2.1 to find the discrete symmetries of the space of these D-dimensional backgrounds. In subsection 4.2.3, we explore the group structure of these symmetries, and find a group isomorphic to $O(d,d,\mathbf{Z})$, as well as a simple expression for its action on the backgrounds [147], analogous to the flat case (see section 2.4). In subsection 4.2.4, we focus on the dilaton and its transformations.

4.2.1 σ-Models Independent of d Coordinates as Quotients

Following [147], we start with a CFT with d Abelian chiral currents J^i and anti-chiral currents $\bar{J}^i$. The parent action is

$$\begin{aligned} S_{D+d} &= S_1 + S_a + S[x]\,, \\ S_1 &= \frac{1}{2\pi}\int d^2z \left[\partial\theta_1^i\bar{\partial}\theta_1^i + \partial\theta_2^i\bar{\partial}\theta_2^i + 2\Sigma_{ij}(x)\partial\theta_2^i\bar{\partial}\theta_1^j + \Gamma^1_{ai}(x)\partial x^a\bar{\partial}\theta_1^i + \Gamma^2_{ia}(x)\partial\theta_2^i\bar{\partial}x^a\right] \\ S_a &= \frac{1}{2\pi}\int d^2z \left[\partial\theta_1^i\bar{\partial}\theta_2^i - \partial\theta_2^i\bar{\partial}\theta_1^i\right] \\ S[x] &= \frac{1}{2\pi}\int d^2z \left[\Gamma_{ab}(x)\partial x^a\bar{\partial}x^b - \frac{1}{4}\Phi(x)R^{(2)}\right], \end{aligned} \tag{4.2.1}$$

where $i,j = 1,\ldots,d$ and $a,b = d+1,\ldots,D$, and $\Sigma_{ij}, \Gamma^1_{ai}, \Gamma^2_{ia}, \Gamma_{ab}$ are components of arbitrary x-dependent matrices, such that, together with the dilaton Φ, the worldsheet theory described by the action S_{D+d} is conformal.

The antisymmetric term S_a is (locally) a total derivative, and therefore, may give only topological contributions, depending on the periodicity of the coordinates θ. We define

$$\theta^i = \theta_2^i - \theta_1^i\,, \qquad \tilde{\theta}^i = \theta_1^i + \theta_2^i\,, \tag{4.2.2}$$

and specify their periodicity:

$$\theta^i \equiv \theta^i + 2\pi\,, \qquad \tilde{\theta}^i \equiv \tilde{\theta}^i + 2\pi\,. \tag{4.2.3}$$

In these coordinates, S_a becomes

$$S_a = \frac{1}{2\pi}\int d^2z\,\frac{1}{2}\Big[\partial\tilde{\theta}^i\bar{\partial}\theta^i - \partial\theta^i\bar{\partial}\tilde{\theta}^i\Big]\,, \tag{4.2.4}$$

which takes half-integer values, and therefore, contributes to the path-integral.

The action S_{D+d} (4.2.1) is invariant under the $U(1)^d_L \times U(1)^d_R$ affine symmetry generated by chiral currents J^i and anti-chiral currents $\bar{J}^i$ given by

$$\begin{aligned} J^i &= \partial\theta^i_1 + \Sigma_{ji}\partial\theta^j_2 + \frac{1}{2}\Gamma^1_{ai}\partial x^a \\ &= \frac{1}{2}\Big[-(I-\Sigma)_{ji}\partial\theta^j + (I+\Sigma)_{ji}\partial\tilde{\theta}^j + \Gamma^1_{ai}\partial x^a\Big]\,, \\ \bar{J}^i &= \bar{\partial}\theta^i_2 + \Sigma_{ij}\bar{\partial}\theta^j_1 + \frac{1}{2}\Gamma^2_{ia}\bar{\partial}x^a \\ &= \frac{1}{2}\Big[(I-\Sigma)_{ij}\bar{\partial}\theta^j + (I+\Sigma)_{ij}\bar{\partial}\tilde{\theta}^j + \Gamma^2_{ia}\bar{\partial}x^a\Big]\,. \end{aligned} \tag{4.2.5}$$

Let us now gauge the d anomaly-free axial combinations of the symmetries (4.2.5) by minimal coupling $\partial\tilde{\theta}^i \to \partial\tilde{\theta}^i + A^i$, $\bar{\partial}\tilde{\theta}^i \to \bar{\partial}\tilde{\theta}^i + \bar{A}^i$. (Other options are generated by discrete symmetries discussed later.) The gauged action is

$$S_{gauged} = S_{D+d} + \frac{1}{2\pi}\int d^2z\,\Big[A^i\bar{J}^i + \bar{A}^i J^i + \frac{1}{2}A^i\bar{A}^j(I+\Sigma)_{ij}\Big]\,. \tag{4.2.6}$$

Integrating out the gauge fields $A^i, \bar{A}^i$ gives:

$$\begin{aligned} S_D &= \frac{1}{2\pi}\int d^2z\,\Big[\mathcal{E}_{IJ}(x)\partial X^I\bar{\partial}X^J - \frac{1}{4}\phi(x)R^{(2)}\Big] \\ &= \frac{1}{2\pi}\int d^2z\,\Big[E_{ij}(x)\partial\theta^i\bar{\partial}\theta^j + F^2_{ia}(x)\partial\theta^i\bar{\partial}x^a + F^1_{ai}(x)\partial x^a\bar{\partial}\theta^i \\ &\qquad + F_{ab}(x)\partial x^a\bar{\partial}x^b - \frac{1}{4}\phi(x)R^{(2)}\Big]\,. \end{aligned} \tag{4.2.7}$$

Notice that S_D is independent of $\tilde{\theta}$, as expected. In eq. (4.2.7)

$$\{X^I | I = 1\ldots D\} = \{\theta^i, x^a \;| i = 1\ldots d, a = d+1\ldots D\}.$$

In the first line in (4.2.7), we have combined the background fields into a $D\times D$ background matrix $\mathcal{E}$

$$\mathcal{E}_{IJ} = \mathcal{G}_{IJ} + \mathcal{B}_{IJ} = \begin{pmatrix} E_{ij} & F^2_{ib} \\ F^1_{aj} & F_{ab}\end{pmatrix}\,. \tag{4.2.8}$$

Here $\mathcal{G}$ and $\mathcal{B}$ denote the symmetric part and antisymmetric part of $\mathcal{E}$, respectively. The four block components in (4.2.8) are

$$E_{ij} = (I-\Sigma)_{ik}(I+\Sigma)^{-1}_{kj}\,, \tag{4.2.9}$$

and

$$F^2_{ia} = (I+\Sigma)^{-1}_{ij}\Gamma^2_{ja}\,, \quad F^1_{ai} = -\Gamma^1_{aj}(I+\Sigma)^{-1}_{ji}\,, \quad F_{ab} = \Gamma_{ab} - \frac{1}{2}\Gamma^1_{ai}(I+\Sigma)^{-1}_{ij}\Gamma^2_{jb}\,. \tag{4.2.10}$$

The D-dimensional background $\mathcal{E}(x)$ is independent of the d coordinates θ^i. It is the most general σ-model with d commuting, compact Abelian symmetries, since for *any* such background the relations (4.2.9, 4.2.10) can be inverted to solve for (Σ, Γ) in terms of (E, F). Moreover, if the original model S_{D+d} (4.2.1) is conformal, then S_D is conformal as well [246],[24] with a dilaton field

$$\phi = \Phi + \log\det(I+\Sigma)\,. \tag{4.2.11}$$

This relation is also invertible. Therefore, a σ-model of a D-dimensional background, which is independent of d coordinates, can be described as a quotient of a worldsheet theory in a $(D+d)$-dimensional background with d chiral currents and d anti-chiral currents.

Finally, if the theory in D-dimensional background is conformally invariant, then the parent theory in $(D+d)$-dimensional background is conformal as well [246], and hence, *any* CFT with a background that is independent of d coordinates can be described as a quotient of a CFT with a $(D+d)$-dimensional background.

4.2.2 Discrete Symmetries

The construction in the previous subsection allow us to understand (some of) the global structure in the moduli space of string vacua in curved backgrounds that are independent of d coordinates. Here we will study discrete symmetries that relate different backgrounds $(E(x), F(x), \phi(x))$ (4.2.7) which correspond to the same CFT.

We first discuss transformations of $(E(x), F(x), \phi(x))$ that follow from manifest symmetries of the action S_{D+d} (4.2.1). We then combine them with transformations that are manifest symmetries of S_D (4.2.7) to find a discrete symmetry group isomorphic to $O(d,d,\mathbf{Z})$.

The S_{D+d} theory is invariant under the coordinate transformations

$$\theta_1 \to O_1\theta_1\,, \qquad \theta_2 \to O_2\theta_2\,, \tag{4.2.12}$$

together with the transformations of the background

$$\Sigma \to O_2\Sigma O_1^t\,, \qquad \Gamma^1 \to \Gamma^1 O_1^t\,, \qquad \Gamma^2 \to O_2\Gamma^2\,, \qquad \Gamma \to \Gamma\,, \tag{4.2.13}$$

where

$$O_1, O_2 \in O(d,\mathbf{Z})\,, \qquad \frac{1}{2}(O_1 \pm O_2)_{ij} \in \mathbf{Z}\,. \tag{4.2.14}$$

[24] Although S_D is conformal only to one loop, higher-order corrections to the background that give an *exact* CFT with a D-dimensional background exist [246, 147, 197]; these corrections come from the integration measure.

Here $O(d,\mathbf{Z})$ is the group of matrices O with integer entries satisfying $OO^t = I$.

These symmetries can be found as follows: The action S_{D+d} is invariant, up to total derivatives, under $O(d,\mathbf{R}) \times O(d,\mathbf{R})$ acting on (θ_1, θ_2) as in (4.2.12), together with (4.2.13) for the backgrounds. The periodic coordinates $\theta, \tilde{\theta}$ (4.2.2) transform as:

$$\begin{pmatrix} \theta \\ \tilde{\theta} \end{pmatrix} \to \frac{1}{2} \begin{pmatrix} O_1 + O_2 & O_1 - O_2 \\ O_1 - O_2 & O_1 + O_2 \end{pmatrix} \begin{pmatrix} \theta \\ \tilde{\theta} \end{pmatrix} \tag{4.2.15}$$

To preserve the periodicities of $\theta, \tilde{\theta}$ (4.2.3), the condition (4.2.14) must be satisfied. In particular, this implies $O_1, O_2 \in O(d,\mathbf{Z})$.

A total derivative comes from the transformation of S_a (4.2.4), and is

$$S_a[O_1\theta_1, O_2\theta_2] - S_a[\theta_1,\theta_2] = \frac{1}{2\pi}\int d^2z \left[M_{ij}(\partial\tilde{\theta}^i\bar{\partial}\tilde{\theta}^j - \partial\theta^i\bar{\partial}\theta^j) + N_{ij}(\partial\theta^i\bar{\partial}\tilde{\theta}^j - \partial\tilde{\theta}^i\bar{\partial}\theta^j)\right], \tag{4.2.16}$$

where

$$M = \frac{1}{4}(O_1^t - O_2^t)(O_1 + O_2)\,, \qquad N = \frac{1}{4}(O_1^t - O_2^t)(O_1 - O_2)\,. \tag{4.2.17}$$

The condition (4.2.14) implies that $M_{ij}, N_{ij} \in \mathbf{Z}$, and hence the total derivative is an integer, and does not contribute to the path integral. This concludes the proof that (4.2.12, 4.2.13) are symmetries of the action S_{D+d}.

The transformations of the background (4.2.13) induce transformations of the background $(E(x), F(x), \phi(x))$ (4.2.9, 4.2.10, 4.2.11) in S_D (4.2.7):

$$E \to E' = (I - O_2\Sigma O_1^t)(I + O_2\Sigma O_1^t)^{-1} \tag{4.2.18}$$

$$\begin{aligned} F^1 &\to F^{1\prime} = -\Gamma^1(O_1 + O_2\Sigma)^{-1} \\ F^2 &\to F^{2\prime} = (O_2^t + \Sigma O_1^t)^{-1}\Gamma^2 \\ F &\to F' = \Gamma - \tfrac{1}{2}\Gamma^1(O_2^t O_1 + \Sigma)^{-1}\Gamma^2 \end{aligned} \tag{4.2.19}$$

$$\phi \to \phi' = \Phi + \log\det(I + O_2\Sigma O_1^t)\,. \tag{4.2.20}$$

These transformations can be rewritten as

$$E' = \Big[(O_1 + O_2)E + (O_1 - O_2)\Big]\Big[(O_1 - O_2)E + (O_1 + O_2)\Big]^{-1} \tag{4.2.21}$$

$$\begin{aligned} F^{1\prime} &= 2F^1\Big[(O_1 - O_2)E + (O_1 + O_2)\Big]^{-1} \\ F^{2\prime} &= \frac{1}{2}\Big[(O_1 + O_2) - E'(O_1 - O_2)\Big]F^2 \\ F' &= F - F^1\Big[(O_1 - O_2)E + (O_1 + O_2)\Big]^{-1}(O_1 - O_2)F^2 \end{aligned} \tag{4.2.22}$$

$$\phi' = \phi + \frac{1}{2}\log\Big[\frac{\det \mathcal{G}}{\det \mathcal{G}'}\Big] = \phi + \frac{1}{2}\log\Big[\frac{\det G}{\det G'}\Big]\,, \tag{4.2.23}$$

where $\mathcal{G}$ is the background metric as defined in (4.2.8) and G is the symmetric part of E (4.2.9). We discuss the dilaton transformation (4.2.23) in detail in subsection 4.2.4.

The transformations (4.2.12, 4.2.13) induce a non-trivial action on the currents (4.2.5), and hence, S'_D (S_D with a transformed background (E', F', ϕ')) is derived from S_{D+d} (4.2.1) by a different quotient. For example, some symmetries simply change the sign of a J^i without changing $\bar{J}^i$; this corresponds to a *vector* gauging (as opposed to an axial gauging) of the i'th $U(1)$. We thus refer to such symmetries as *axial-vector duality* [196] (and we show that these are exact symmetries in section 4.8).

We now consider the additional transformations that are manifest symmetries of the action S_D itself. The first are integer "Θ"-parameters that shift E:

$$\begin{aligned} &E_{ij} \to E_{ij} + \Theta_{ij}\,, \qquad \Theta_{ij} = -\Theta_{ji} \in \mathbf{Z} \\ &F^1 \to F^1\,, \qquad F^2 \to F^2\,, \qquad F \to F\,. \end{aligned} \tag{4.2.24}$$

We refer to the group generated by these transformations as $\Theta(\mathbf{Z})$. These are obviously symmetries, as they shift the action S_D (4.2.7) by an integer, and therefore, do not change the path integral.

The second type of transformations are given by homogeneous transformations of E, F^1, F^2 under $A \in GL(d, \mathbf{Z})$:

$$E \to A^t E A\,, \qquad F^2 \to A^t F^2\,, \qquad F^1 \to F^1 A\,, \qquad F \to F\,. \tag{4.2.25}$$

These are obviously symmetries of the theory described by S_D, as they generate a change of basis in the space of θ's that preserves their periodicities.

Neither the $\Theta(\mathbf{Z})$ nor the $GL(d, \mathbf{Z})$ transformations affect the dilaton, since they do not change the integration measure of the path integral; consequently, they are symmetries of the exact CFTs, and receive no higher-order corrections.

The group generated by all the symmetries discussed is isomorphic to $O(d, d, \mathbf{Z})$. A natural embedding of $O(d, d, \mathbf{Z})$ in $O(D, D, \mathbf{Z})$ acts on the background $\mathcal{E}$ by fractional linear transformations, analogous to the flat case and as explained in the following subsection.

4.2.3 The Action of $O(d, d, \mathbf{Z})$

We begin by establishing our notation following section 2.4 and refs. [141, 147]. As explained in section 2.4 the group $O(d, d, \mathbf{R})$ can be represented as $(2d \times 2d)$-dimensional matrices g

preserving the bilinear form J:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{4.2.26}$$

where a, b, c, d, I are $(d \times d)$-dimensional matrices, and

$$g^t J g = J \quad \Rightarrow \quad a^t c + c^t a = 0\,, \quad b^t d + d^t b = 0\,, \quad a^t d + c^t b = I\,. \tag{4.2.27}$$

This has an obvious embedding in $O(D, D, \mathbf{R})$ as

$$\hat{g} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}, \tag{4.2.28}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ are $(D \times D)$-dimensional matrices of the form

$$\hat{a} = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{c} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{d} = \begin{pmatrix} d & 0 \\ 0 & I \end{pmatrix} \tag{4.2.29}$$

(here I is the $(D-d) \times (D-d)$-dimensional identity matrix).

We define the action of $\hat{g}$ on $\mathcal{E}$ by fractional linear transformations:

$$\begin{aligned} \hat{g}(\mathcal{E}) =& \mathcal{E}' = (\hat{a}\mathcal{E} + \hat{b})(\hat{c}\mathcal{E} + \hat{d})^{-1} \\ =& \begin{pmatrix} E' & (a - E'c)F^2 \\ F^1(cE+d)^{-1} & F - F^1(cE+d)^{-1}cF^2 \end{pmatrix} \end{aligned}, \tag{4.2.30}$$

where

$$E' = (aE + b)(cE + d)^{-1} \tag{4.2.31}$$

is a fractional linear transformation of E under $O(d, d, \mathbf{R})$. If $\mathcal{E}$ is constant we recover the action of $O(D, D, \mathbf{R})$ on the D-dimensional toroidal background matrix, as discussed in section 2.4.

Let us recall here that the group $O(d, d, \mathbf{R})$ is generated by:

$GL(d, \mathbf{R})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix} \quad \text{s.t.} \quad A \in GL(d, \mathbf{R})\,. \tag{4.2.32}$$

$\Theta(\mathbf{R})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix} \quad \text{s.t.} \quad \Theta = -\Theta^t\,. \tag{4.2.33}$$

Factorized duality:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I - e_1 & e_1 \\ e_1 & I - e_1 \end{pmatrix} \quad \text{s.t.} \quad e_1 = diag(1, 0, \ldots, 0)\,. \tag{4.2.34}$$

The maximal compact subgroup of $O(d, d, \mathbf{R})$ is $O(d, \mathbf{R}) \times O(d, \mathbf{R})$ embedded as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2} \begin{pmatrix} o_1 + o_2 & o_1 - o_2 \\ o_1 - o_2 & o_1 + o_2 \end{pmatrix} \quad \text{s.t.} \quad o_1, o_2 \in O(d, \mathbf{R})\,. \tag{4.2.35}$$

This subgroup includes factorized duality (4.2.34).

We now turn from definitions to the actual symmetries of the CFT. These form an $O(d,d,\mathbf{Z})$ discrete subgroup of $O(d,d,\mathbf{R})$ that acts on the background as above. The elements of the subgroup $O(d,d,\mathbf{Z})$ are given by matrices g of the form (4.2.26, 4.2.27) with *integer* entries.

Just as in the continuous case, the discrete group is generated by $GL(d,\mathbf{Z})$, $\Theta(\mathbf{Z})$, and factorized duality; these are given by (4.2.32, 4.2.33, 4.2.34) with integer entries. The subgroup $O(d,\mathbf{Z})\times O(d,\mathbf{Z})$ is given by the matrices (4.2.35), again with integer entries. Clearly, the $O(d,\mathbf{Z})\times O(d,\mathbf{Z})$ symmetries (4.2.21, 4.2.22), the $\Theta(\mathbf{Z})$ symmetries (4.2.24), and the $GL(d,\mathbf{Z})$ symmetries (4.2.25), that we found in the previous subsection, act on the background by the $O(d,d,\mathbf{Z})\subset O(D,D,\mathbf{Z})$ fractional linear transformations (4.2.30) with the matrices a,b,c,d given by (4.2.35), with $o_{1,2}=O_{1,2}$, (4.2.33), with $\Theta\in\mathbf{Z}$, and (4.2.32), with $A\in GL(d,\mathbf{Z})$, respectively.

These results are compatible with the discrete symmetries of the space of flat D-dimensional toroidal backgrounds [146, 260, 140], described in section 2.4. In that case, the $O(d,d,\mathbf{Z})$ symmetries, described above, are simply a subgroup of the full $O(D,D,\mathbf{Z})$ symmetry group which acts as in (4.2.30), for *any* $\hat{g}\in O(D,D,\mathbf{Z})$. For curved D-dimensional backgrounds, we expect to find some large symmetry group analogous to $O(D,D,\mathbf{Z})$; here we have described its $O(d,d,\mathbf{Z})$ subgroup that is associated with a d-dimensional toroidal isometry of the background.

In the flat case, the fractional linear transformation is an exact map between equivalent backgrounds; in the curved case, in general, one expects higher-order corrections to the transformed background [34, 35, 33, 264, 271]. For $\Theta(\mathbf{Z})$ and $GL(d,\mathbf{Z})$, the transformations are exact; however, factorized duality receives corrections from the path-integral measure. Nevertheless, because the transformation is exact in the $(D+d)$-dimensional model, we know that non-perturbative correction must exist such that factorized duality is exact[25].

We close this subsection with a general remark. The group $O(d,d,\mathbf{R})$ has disconnected components; in particular, factorized duality has $\det=-1$, and hence is not connected to the identity. Therefore, one expects that in general the moduli space is made of several disconnected components mapped into each other by $O(d,d,\mathbf{R})$ transformations. In the flat case the moduli space is connected and this phenomenon does not occurs. However, in other cases one may get various disconnected components of backgrounds with different topologies [26]. This issue will be addressed later.

[25] $N=4$ supersymmetry can protect duality transformations, and there are examples where the one-loop transformations are exact even in curved backgrounds. Exact duality was also described for superstrings and $(1,1)$ heterotic backgrounds corresponding to cosets in [145], and in some examples of bosonic string backgrounds [139, 272, 200].

[26] Factorized duality is similar to mirror symmetry [159], which also identifies two (possibly) disconnected components of moduli space corresponding to backgrounds with different topologies. For $N=2$ toroidal (orbifold) backgrounds, mirror symmetry and factorized duality are identical [150].

4.2.4 The Dilaton

To complete the previous discussion of the transformation of the background under $O(d,d,\mathbf{Z})$, we consider the transformation of the dilaton [34, 136, 140, 5, 271, 147, 251]. This is summarized in eq. (4.2.23), which is derived by proving that the quantity

$$\tilde{\phi} = \phi + \frac{1}{2}\log\det\mathcal{G} \tag{4.2.36}$$

is invariant under $O(d,d,\mathbf{Z})$ transformations [147]. This implies

$$\phi' = \phi + \frac{1}{2}\log\Big(\frac{\det\mathcal{G}}{\det\mathcal{G}'}\Big)\,. \tag{4.2.37}$$

The second equality in (4.2.23) follows from the proof of (4.2.36) given below.

We begin with the identity

$$\begin{pmatrix} G_{ij} & G_{ib} \\ G_{aj} & G_{ab}\end{pmatrix} = \begin{pmatrix} G_{ik} & 0 \\ G_{ak} & I_{ac}\end{pmatrix}\begin{pmatrix} I_{kj} & (G^{-1})_{kl}G_{lb} \\ 0 & G_{cb} - G_{cm}(G^{-1})_{ml}G_{lb}\end{pmatrix} \tag{4.2.38}$$

which implies

$$\det(\mathcal{G}) = \det(G_{ij})\det(G_{ab} - G_{ak}(G^{-1})_{kl}G_{lb})\,. \tag{4.2.39}$$

We next prove that the following two quantities are separately invariant under $O(d,d,\mathbf{Z})$ transformations:

$$\text{the invariant fiber dilaton}: \quad \hat{\phi} = \phi + \frac{1}{2}\log\det(G_{ij})\,, \tag{4.2.40}$$

$$\text{the quotient metric}: \quad G_{ab} - G_{ak}(G^{-1})_{kl}G_{lb}\,. \tag{4.2.41}$$

Geometrically, a D-dimensional space whose metric is independent of d coordinates θ^i can be thought of as a bundle M with fiber coordinates θ. The metric on the fiber is G_{ij}, and hence we refer to $\hat{\phi}$ as the "invariant fiber dilaton." The induced metric on the quotient space $M/\{\theta^i\}$ is the quotient metric (4.2.41).

To prove the invariance of (4.2.40), we consider the action of the generators of $O(d,d,\mathbf{Z})$ separately. The $GL(d,\mathbf{Z})$ and $\Theta(\mathbf{Z})$ transformations trivially leave ϕ and $\det(G_{ij})$ invariant, and hence $\hat{\phi}$ as well. To show invariance under factorized duality, we write $\hat{\phi}$ explicitly in terms of Σ:

$$\begin{aligned}\hat{\phi} &= \Phi + \log\det(I+\Sigma) + \frac{1}{2}\log\det\frac{1}{2}\Big[(I-\Sigma)(I+\Sigma)^{-1} + (I+\Sigma^t)^{-1}(I-\Sigma^t)\Big] \\ &= \Phi + \frac{1}{2}\log\det(I-\Sigma^t\Sigma)\,.\end{aligned} \tag{4.2.42}$$

To get the second equality, we need to split $\log\det(I+\Sigma)$ as $\frac{1}{2}\log\det(I+\Sigma) + \frac{1}{2}\log\det(I+\Sigma^t)$. Under $\Sigma \to O_2\Sigma O_1^t$ (4.2.13), $(I-\Sigma^t\Sigma) \to O_1(I-\Sigma^t\Sigma)O_1^t$, and hence $\hat{\phi}$ is invariant. This completes the proof of the invariance of the fiber dilaton $\hat{\phi}$.

The proof of the invariance of (4.2.41) is entirely parallel; again, the $GL(d,\mathbf{Z})$ and $\Theta(\mathbf{Z})$ transformations trivially leave (4.2.41) unchanged. To prove invariance under factorized duality, we express (4.2.41) explicitly in terms of $\Sigma, \Gamma^1, \Gamma^2, \Gamma$ (see (4.2.9, 4.2.10)):

$$\begin{aligned}
&G_{ab} - G_{ak}(G^{-1})_{kl}G_{lb} \\
&= \Big[\frac{1}{2}(\Gamma+\Gamma^t) - \frac{1}{4}\{\Gamma^1(I+\Sigma)^{-1}\Gamma^2 + \Gamma^{2t}(I+\Sigma^t)^{-1}\Gamma^{1t}\} \\
&\quad + \frac{1}{2}\{\Gamma^1(I+\Sigma)^{-1} + \Gamma^{2t}(I+\Sigma^t)^{-1}\}\{(I-\Sigma)(I+\Sigma)^{-1} + (I+\Sigma^t)^{-1}(I-\Sigma^t)\}^{-1} \\
&\qquad\qquad \times \{(I+\Sigma)^{-1}\Gamma^2 + (I+\Sigma^t)^{-1}\Gamma^{1t}\}\Big]_{ab} \\
&= \Big[\frac{1}{2}(\Gamma+\Gamma^t) + \frac{1}{4}\{\Gamma^1(I-\Sigma^t\Sigma)^{-1}(\Gamma^{1t}+\Sigma^t\Gamma^2) + \Gamma^{2t}(I-\Sigma\Sigma^t)^{-1}(\Gamma^2+\Sigma\Gamma^{1t})\}\Big]_{ab}.
\end{aligned} \tag{4.2.43}$$

The final expression is manifestly invariant under (4.2.13), which completes the proof of the invariance of the quotient metric (4.2.41).

The invariance of $\tilde{\phi}$ in (4.2.36) follows from the invariance of (4.2.40, 4.2.41) together with the identity (4.2.39). This is compatible with results in low-energy effective field theories [144, 225], as discussed in section 2.8, and string field theory [254, 204], and physically implies that the string coupling constant $g_{string}^{-1} =< e^{\tilde{\phi}} >=< \sqrt{\det \mathcal{G}}\, e^{\phi} >$ is invariant under $O(d,d,\mathbf{Z})$. The invariance of (4.2.40, 4.2.41) actually proves that $\sqrt{\det(G_{ij})}\, e^{\phi}$ and the quotient metric (4.2.41) are separately invariant; of course, this holds only for $O(d,d,\mathbf{Z})$, and not for the full $O(D,D,\mathbf{Z})$ in which the former group is embedded.

4.3 The Action of $O(d,d,\mathbf{R})$

In the previous subsection we have discussed the action of the discrete group $O(d,d,\mathbf{Z})$. The *continuous* group $O(d,d,\mathbf{R})$ also maps a conformal background [141, 278, 225, 122, 219], that is independent of d coordinates, onto (the leading order of) a conformal background. Namely, given a conformal background $\mathcal{E}, \phi$ and an element g in $O(d,d,\mathbf{R})$ (4.2.26), then the background $g(\mathcal{E},\phi) = (\mathcal{E}', \phi')$ given by eqs. (4.2.30) and (4.2.37) is the leading order of an exact conformal background.

This is true because not only the discrete groups $\Theta(\mathbf{Z})$ and $GL(d,\mathbf{Z})$, but also $\Theta(\mathbf{R})$ and $GL(d,\mathbf{R})$, transform exact backgrounds to exact backgrounds. Moreover, we have shown that there is a CFT corresponding to an exact background whose leading order is given by acting on the original one with factorized duality, as explained in the previous section. Now, because the full $O(d,d,\mathbf{R})$ group is generated by $\Theta(\mathbf{R})$, $GL(d,\mathbf{R})$ and factorized duality, we conclude that the action of $O(d,d,\mathbf{R})$ on a conformal background with d commuting, compact Abelian symmetries generates the leading order of an exact background.

The moduli space of curved string backgrounds with d Abelian symmetries (generated by

$O(d,d,\mathbf{R})$ rotations) was suggested at first by studies of both low-energy effective actions [141, 89, 143, 144, 225, 122] and worldsheet actions on the sphere [116, 50, 227]. These studies suggested that there is a moduli subspace of backgrounds, independent of d coordinates, isomorphic to $O(d,d,\mathbf{R})/G$. Here G is at least the diagonal subgroup $O(d,\mathbf{R})_{diag}$ of $O(d,\mathbf{R})\times O(d,\mathbf{R})$ (the maximal compact subgroup of $O(d,d,\mathbf{R})$). (Needless to say, for flat backgrounds $G=O(d,\mathbf{R})\times O(d,\mathbf{R})$ [236, 238]). This is consistent with results from string field theory [254]. This moduli subspace is only correct when the d coordinates have the topology of a torus. Though in general the local structure of the moduli space is unknown, when the background is independent of d periodic coordinates, we can still identify a discrete symmetry group that acts on the moduli space, as described in the previous section.

The method of $O(d,d,\mathbf{R})$ transformations is useful to generate a large class of curved string backgrounds. Generating solutions in this way was discussed extensively in the literature. For example, in refs. [255, 166, 179, 173, 147, 256, 190, 184] $O(d,d,\mathbf{R})$ rotations were used in order to generate many kinds of (charged) black objects (black holes, black strings etc.); in refs. [225, 122, 142, 123, 192, 2, 193] this method was used to generate cosmological solutions. In sections 4.5, 4.6 we will describe examples of black-object solutions, and in section 4.7 we will describe examples of cosmological string backgrounds with Abelian symmetries.

4.4 The Heterotic String

In this section, we discuss the explicit form of the discrete symmetries of the heterotic string following ref. [147]. This form can be shown to hold to leading order in α' for $N=(1,1)$ cosets and their deformations following refs. [145, 139]. Its validity for the general case was conjectured in ref. [147], by requiring compatibility with the flat limit [146, 260], described in section 2.5, and with the bosonic case. We start with a *curved* heterotic background, which we assume is a consistent, conformally invariant, heterotic string theory, with action:

$$\begin{aligned} S_{heterotic} =& \frac{1}{2\pi}\int d^2z \Big[\mathcal{E}_{IJ}(x)\partial X^I\bar\partial X^J + \mathcal{A}_{IA}(x)\partial X^I\bar\partial Y^A \\ &+ E_{AB}\partial Y^A\bar\partial Y^B - \frac{1}{4}\phi(x)R^{(2)} + (\text{fermionic terms})\Big]\,, \end{aligned} \tag{4.4.1}$$

and with the constraints that the Y^A are covariantly chiral bosons: $\partial Y^A + \mathcal{A}^A_I\partial X^I = 0$. As before $\{X^I\}=\{\theta^i, x^a | i=1\ldots d \text{ and } a=d+1\ldots D\}$. In addition, we have d_{int} internal chiral bosons Y^A: $A=1\ldots d_{int}$. In flat space we have $d_{int}=16$, but more generally, we may find other solutions [71, 224]. The space-time background is given by ($\mathcal{E}$,ϕ), as in the bosonic case (4.2.8, 4.2.11), and, in addition, the gauge field $\mathcal{A}$. We assume that this curved background is independent of the d coordinates θ^i. The constant internal background is

$$E_{AB} = G_{AB} + B_{AB}\,, \tag{4.4.2}$$

where G_{AB} is the metric on the internal lattice (one half the Cartan matrix of the internal symmetry group when the lattice is the root lattice of a group) and B_{AB} is its antisymmetrization [94], *i.e.*, E_{AB} is upper triangular. In the space-time supersymmetric flat case, the symmetry group is $E_8 \times E_8$.[27] In curved space, this group is in general different [71, 224].

Following section 2.5 [146], the expected symmetry group is isomorphic to $O(d, d+d_{int}, \mathbf{Z})$ embedded in $O(D, D+d_{int}, \mathbf{Z})$ as in section 2.5, and acting by fractional linear transformations on the $(D+d_{int}) \times (D+d_{int})$ dimensional matrix

$$\Xi(x) = \begin{pmatrix} \mathcal{E}_{IJ} + \frac{1}{4}\mathcal{A}_{IA}(G^{-1})_{AB}\mathcal{A}_{JB} & \mathcal{A}_{IA} \\ 0 & E_{AB} \end{pmatrix}. \tag{4.4.3}$$

Here $(G^{-1})_{AB}$ is the inverse of G_{AB} in (4.4.2). The matrix Ξ is the embedding of the heterotic background into a bosonic $(D+d_{int})$-dimensional background, and the group $O(D, D+d_{int})$ is the subgroup of $O(D+d_{int}, D+d_{int})$ that preserves the form (4.4.3). Note that the space-time metric $\mathcal{G}_{IJ}$ (the symmetric part of $\mathcal{E}_{IJ}$ in (4.4.3)) is the quotient metric of the $(D+d_{int})$-dimensional space modulo $\{Y^A\}$ (here, G_{AB} is the fiber metric). This leads to a simple expression for the transformation of the dilaton:

$$\phi' = \phi + \frac{1}{2}\log\left(\frac{\det \mathcal{G}}{\det \mathcal{G}'}\right); \tag{4.4.4}$$

note that this is independent of the gauge fields.

4.5 Application I: the D=2 Black Hole Duality

In this section, we discuss the first nontrivial example of curved-space duality, namely, a $D = 2$ background that is independent of one coordinate ($d = 1$). Such a background can be derived as a gauged WZNW model $SU(2)/U(1)$ or $SL(2, \mathbf{R})/U(1)$. The exact conformal field theory corresponding to an $SL(2, \mathbf{R})/U(1)$ gauged WZNW model [20] describes a black hole in two dimensional space-time [285]. We will describe the duality of the $D = 2$ black hole [137, 83] in some detail to examine explicitly the procedure presented in section 4.2 for this simple example. Moreover, we will discuss some of its interesting consequences, namely: interchanging the horizon with the singularity, and taking seriously the space-time region beyond the singularity. This example also illustrates the possibility to generate D-dimensional curved backgrounds, independent of d coordinates, from G/H coset CFTs and their duals.

The G_k WZNW model and its gauging

At first, let us describe briefly a general WZNW model on a group manifold G at level k (we call it the G_k WZNW model), and its gauging. The action of the WZNW model is [282]

$$S[g] = \frac{k}{4\pi} I[g] + \frac{k}{6\pi}\Gamma[g], \tag{4.5.1}$$

[27] The $Spin(32)/\mathbf{Z}_2$ string can be described as the $E_8 \times E_8$ string with a particular gauge field background [132, 236, 238].

$$I[g] = \int_\Sigma d^2\sigma \sqrt{h} h^{ij} \mathrm{Tr}(e_i e_j), \qquad \Gamma[g] = \int_{B,\, \partial B = \Sigma} d^3\sigma \epsilon^{ijk} \mathrm{Tr}(e_i e_j e_k), \tag{4.5.2}$$

where Σ is a worldsheet with coordinates σ_i and metric h_{ij}, and

$$e_i \equiv g^{-1}\partial_i g, \tag{4.5.3}$$

where the field g is a map from the worldsheet to the group manifold G; it can be taken to be a matrix in the fundamental representation of G. The trace in (4.5.2) is an invariant form on the Lie algebra of G, and the Wess-Zumino term, $\Gamma[g]$, produces an antisymmetric background on the worldsheet Σ, after integration over the third coordinate of B. Explicitly, in the matrix representation T for the Lie algebra,

$$[T_a, T_b] = f_{ab}^c T_c, \qquad \mathrm{Tr}(T_a T_b) = \eta_{ab}, \qquad a, b, c = 1, ..., \dim G, \tag{4.5.4}$$

and parametrizing $g(X) \in G$ in a set of coordinates $X^A(\sigma_i)$, $A = 1, ..., \dim G$, one finds

$$S[X] = \frac{1}{4\pi} \int d^2\sigma \left[\sqrt{h} h^{ij} G_{AB}(X) \partial_i X^A \partial_j X^B + \epsilon^{ij} B_{AB}(X) \partial_i X^A \partial_j X^B \right], \tag{4.5.5}$$

where $G_{AB}(X)$, $B_{AB}(X)$ are given by the left-invariant vielbeins, $e_A(X)$,

$$\begin{aligned} e_A \equiv g^{-1}\partial_A g = e_A^a T_a, \qquad & \partial_A e_B^c - \partial_B e_A^c = e_A^a e_B^b f_{ab}^c, \\ G_{AB} &= k \eta_{ab} e_A^a e_B^b, \\ k \mathrm{Tr}\left(e_A [e_B, e_C]\right) &= \partial_A B_{BC} + \partial_B B_{CA} + \partial_C B_{AB}. \end{aligned} \tag{4.5.6}$$

The WZNW model can serve as a starting point to generate other CFTs. From an algebraic point of view this was done in what is termed the "coset model" [21, 154]. Given a G_k WZNW model with central charge c_G, a subgroup $H_{k'}$ (of appropriate level k' and central charge c_H) could be used to generate a new Virasoro algebra of central charge $c_{G/H} = c_G - c_H$.

A Lagrangian formulation for this method was found by gauging an anomaly-free subgroup $H \subset G$ [233, 22]. The Lagrangian does not contain a kinetic term for the gauge fields. In the absence of these terms, the gauge theory is also conformal. Modular invariance of the G_k model ensures modular invariance of the coset. The coset Lagrangian reproduces the central charge $c_{G/H}$, and the appropriate spectrum. The gauged WZNW model has as basic fields the matter ones, appearing in the G_k model, and the gauge fields. One may rewrite the gauged model in terms of non-linear σ-model variables. This is done by integrating out the gauge fields [20]. To one loop the backgrounds need to be supplemented by a dilaton field [285].

4.5.1 The Euclidean $SL(2,\mathbf{R})/U(1)$ Black Hole and its Dual

To write the action for the gauged $SL(2,\mathbf{R})$ WZNW model we will parametrize the group manifold by "Euler angles." Following the notation in [83] we introduce real coordinates x, θ_1, θ_2, and write $g \in SL(2,\mathbf{R})$ as

$$g = e^{i\theta_1 \sigma_3} e^{x\sigma_2} e^{i\theta_2 \sigma_3}, \tag{4.5.7}$$

with σ_a the Pauli matrices. The coordinates θ_1, θ_2 are such that $\theta = \theta_2 - \theta_1$ and $\tilde{\theta} = \theta_1 + \theta_2$ have periodicity 2π, and $0 \leq x < \infty$. This parametrization is convenient for describing the CFT of the Euclidean $SL(2,\mathbf{R})/U(1)$ coset; for the Lorentzian case one should simply replace $\sigma_3 \to \sigma_1$ and $\theta_{1,2} \to t_{1,2}$.

Inserting (4.5.7) into (4.5.1), one finds that in these target space coordinates, and in complex worldsheet coordinates, the $SL(2,\mathbf{R})_k$ WZNW action is

$$S_{WZNW}[x,\theta_1,\theta_2] = \frac{-k}{2\pi}\int d^2z(-\partial x\bar{\partial}x + \partial\theta_1\bar{\partial}\theta_1 + \partial\theta_2\bar{\partial}\theta_2 + 2\cosh 2x\ \partial\theta_2\bar{\partial}\theta_1). \tag{4.5.8}$$

This action has the same structure as $S_1 + S[x]$ in (4.2.1) with $\Sigma = \cosh 2x$, $\Gamma = -1$ and $\Gamma^1 = \Gamma^2 = 0$ (except for an overall k factor[28], and the antisymmetric term S_a that is understood to be present).

The action (4.5.8) is invariant under the $U(1)_L \times U(1)_R$ affine symmetry generated by the currents

$$J = -k(\partial\theta_1 + \cosh 2x\ \partial\theta_2), \qquad \bar{J} = -k(\bar{\partial}\theta_2 + \cosh 2x\ \bar{\partial}\theta_1). \tag{4.5.9}$$

These currents are the same as in (4.2.5) up to the overall k factor discussed above. Therefore, the gauged WZNW action takes the form (4.2.6)

$$\begin{aligned} S_{gauged} &= S_{WZNW}[x,\theta_1,\theta_2] \\ &\quad + \frac{k}{2\pi}\int d^2z[-A(\bar{\partial}\theta_2 + \cosh 2x\ \bar{\partial}\theta_1) - \bar{A}(\partial\theta_1 + \cosh 2x\ \partial\theta_2) + \frac{1}{2}A\bar{A}(1+\cosh 2x)]. \end{aligned} \tag{4.5.10}$$

Integrating out the gauge fields and taking $k \to -k$ gives:

$$S[x,\theta] = \frac{1}{2\pi}\int d^2z[k(\partial x\bar{\partial}x + \tanh^2 x\partial\theta\bar{\partial}\theta) - \frac{1}{4}\phi(x)R^{(2)}], \tag{4.5.11}$$

where θ is defined in (4.2.3), the $\tanh^2 x$ is found by using eq. (4.2.9), and the dilaton is derived by eq. (4.2.11):

$$\phi(x) = \phi_0 + \log(\cosh^2 x). \tag{4.5.12}$$

Up to $1/k$ corrections, the metric in (4.5.11) and the dilaton field in (4.5.12) satisfy the one-loop beta-function equation [92, 222] of ref. [39]

$$R_{ab} = D_a D_b \phi, \tag{4.5.13}$$

where R_{ab} is the Ricci tensor of the target space.

[28] The generalization of the procedure of section 4.2 to the case when there is an overall k factor in front of the action (4.2.1) can be done straightforwardly; we have presented only the $k = 1$ case in section 4.2 because this is already sufficient to generate *all* the D-dimensional backgrounds with d Abelian symmetries.

The central charge of the $SL(2,\mathbf{R})/U(1)$ model is

$$c = 2 + \frac{6}{k-2} = 2 + 3\epsilon^2 + \mathcal{O}(\epsilon^4), \tag{4.5.14}$$

where

$$\epsilon = \sqrt{2/k}. \tag{4.5.15}$$

It is convenient to absorb the overall factor k in the coordinate x. Defining $r = x/\epsilon$, one finds the action

$$S[r,\theta] = \frac{1}{2\pi}\int d^2\sigma\sqrt{h}h^{ij}\left(\partial_i r\partial_j r + \frac{\tanh^2\epsilon r}{\epsilon^2}\partial_i\theta\partial_j\theta\right) - \frac{1}{8\pi}\int d^2\sigma\sqrt{h}\phi(r,\theta)R^{(2)}, \tag{4.5.16}$$

where

$$\phi(r) = \phi_0 + \log\cosh^2\epsilon r. \tag{4.5.17}$$

For completeness and clarification we wrote the action in (4.5.16) with the worldsheet coordinates σ_i (2.5.2) and worldsheet metric h_{ij} appearing explicitly, as in section 2.1.

The action in (4.5.16) describes a σ-model on a background with scalar curvature

$$\mathcal{R}(r) = \frac{4\epsilon^2}{\cosh^2\epsilon r}, \tag{4.5.18}$$

and a metric described by the line element

$$ds^2 = dr^2 + \frac{\tanh^2\epsilon r}{\epsilon^2}d\theta^2. \tag{4.5.19}$$

This background defines a semi-infinite cigar (see figure 1.F) with radius

$$R_0(r) = \frac{\tanh\ \epsilon r}{\epsilon}. \tag{4.5.20}$$

This target space can be interpreted [285] as a Euclidean black hole (see for example [280, 226]) with a temperature $1/\beta$, which is given by the inverse of the radius at infinity, namely

$$\beta = \frac{1}{\epsilon}. \tag{4.5.21}$$

We now turn to the target space duality transformation of the Euclidean black hole [137, 83, 23]. Let us consider a background of the type

$$G_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & R^2(r) \end{pmatrix}. \tag{4.5.22}$$

As it was shown in section 4.2, the metric in (4.5.22) and its appropriate dilaton (that solve eq. (4.5.13)) transform under target space duality by

$$R(r) \to R'(r) = \frac{1}{R(r)}, \qquad \phi \to \phi' = \phi + \log R(r)^2. \tag{4.5.23}$$

This symmetry also survives when r is a time-like coordinate [264]. In the limit of a constant background, the duality transformation in (4.5.23) coincides with the one in (2.2.11), describing the simple circle duality $R \to 1/R$ [194, 247].

In the case of a Euclidean black hole $R(r)$ is given in eq. (4.5.20), and therefore, the dual action takes the form

$$S'[r,\theta] = \frac{1}{2\pi}\int d^2\sigma\sqrt{h}h^{ij}(\partial_i r\partial_j r + \epsilon^2\coth^2\epsilon r\ \partial_i\theta\partial_j\theta) - \frac{1}{8\pi}\int d^2\sigma\sqrt{h}\phi'(r,\theta)R^{(2)}, \qquad (4.5.24)$$

where

$$\phi' = \log\sinh^2\epsilon r + \phi_0 - \log\epsilon^2. \qquad (4.5.25)$$

The dual target space has the form of an infinite trumpet (see figure 1.F), with an r-dependent radius

$$R'_0(r) = \epsilon\coth\epsilon r. \qquad (4.5.26)$$

When r approaches 0 the radius diverges, while at the limit $r \to \infty$ the radius approaches ϵ asymptotically. The "temperature" of the "dual black hole" is therefore

$$\frac{1}{\beta'} = \beta = \frac{1}{\epsilon}. \qquad (4.5.27)$$

It is remarkable that while the Euclidean black hole background has a trivial fundamental group, the dual model has $\pi_1(\text{trumpet}) = \mathbf{Z}$.

To clarify the property of the symmetry, let us discuss the $\epsilon \to 0$ limit. The line element (4.5.19) in this limit describes a flat open two dimensional space in polar coordinates

$$ds^2_{\epsilon\to 0} = dr^2 + r^2 d\theta^2. \qquad (4.5.28)$$

The dominant states in the Hilbert space for small ϵ are therefore "momentum" states. Those may be interpreted as particles sent towards the black hole. The target space of the dual model, although complicated in terms of the r,θ coordinates, for which

$$(ds')^2_{\epsilon\to 0} = dr^2 + r^{-2}d\theta^2, \qquad \phi = \phi_0 + \log r^2, \qquad (4.5.29)$$

is similar, as a CFT, to a two dimensional torus which partly shrinks to a point. The dominant states in the Hilbert space for small ϵ are now "winding modes." Such states do not have an analog in standard general relativity or in Kaluza-Klein theory; they are a direct consequence of a string wrapping around circles while propagating in a compact target space. The duality transformation is a quantum symmetry of string vacua which interchanges momentum modes with winding modes[29].

[29] In ref. [197] the duality of $D = 2$ flat space in polar coordinates was further studied in the mini-superspace approximation. This is an attempt to understand duality in case the target space M has no $\pi_1(M)$.

To lowest order in worldsheet perturbation theory, the graviton-dilaton system can be described by a space-time effective action

$$\begin{aligned} L &= \int d^2X\sqrt{G}e^{\phi}(\mathcal{R}+G^{ab}\partial_a\phi\partial_b\phi+\frac{8}{k-2}) \\ &= \int d^2X Re^{\phi}\left(-\frac{\partial^2R^2}{R^2}+2\left(\frac{\partial R}{R}\right)^2+\partial\phi\partial\phi-\frac{8}{k-2}\right). \end{aligned} \tag{4.5.30}$$

The additive constant $8/(k-2)$ originates from the familiar $D-26$ of the bosonic string. The action in (4.5.30) is invariant under duality: $R\to R^{-1}$, $\phi\to\phi+\log R^2$. Thus, the graviton and dilaton field equations derived from this action are duality invariant as well.

4.5.2 The Lorentzian $SL(2,\mathbf{R})/U(1)$ Black Hole and its (Self-)Dual

We now turn to the study of the analytic continuation of the black hole to a Lorentz signature. Let us first discuss the analytic continuation and the Kruskal extension [280, 226] of the original black hole following [285, 137]. Naively, one simply sets $\theta=it$ [30], whereupon the line element (4.5.19) is changed to

$$ds^2=dr^2-\frac{\tanh^2\epsilon r}{\epsilon^2}\,dt^2. \tag{4.5.31}$$

This metric seems to have a singularity at $r=0$, but this must be purely a coordinate singularity, since the scalar curvature (4.5.18) is regular at $r=0$. The analytic continuation past the coordinate singularity can be found by an extension, similar to the one used for the Schwarzschild solution [280, 226]. Defining the u,v coordinates

$$v=\sinh\epsilon r\;e^t,\qquad u=-\sinh\epsilon r\;e^{-t}, \tag{4.5.32}$$

one gets

$$ds^2=-\frac{1}{\epsilon^2}\frac{dudv}{1-uv}. \tag{4.5.33}$$

The dilaton field is given by (4.5.17)

$$\phi=\phi_0+\log(1-uv). \tag{4.5.34}$$

This solution is similar to that of a black hole in Kruskal-Szekeres coordinates as discussed in [285, 222].

A space-time diagram for the u,v coordinates is shown in figure 4.A. This diagram has the same causal structure as the corresponding four-dimensional black hole. In an astrophysical black hole, which forms from a spherically symmetric collapsing star, some of the regions are

[30] Instead of obtaining the Lorentz signature black hole by a formal analytic continuation from a Euclidean space, one can obtain it directly as a conformal field theory by gauging a different $U(1)$ subgroup of $SL(2,\mathbf{R})$ [285].

missing, but one still has parts of regions I,II and V. The causal structure of the extended space-time is easily seen from the diagram in figure 4.A since, by construction, the $u = 0$ and $v = 0$ lines (the horizons) are null geodesic lines. Region I in figure 4.A corresponds to the original "Lorentzian cigar" ($\sinh^2 \epsilon r > 0$ i.e. $uv < 0$), and represents the space-time region outside the past and future horizons. Region II represents the space-time between the future horizon and the future singularity. In general relativity a signal sent from region V (where time direction goes sideways) cannot cross the horizon towards an observer in region I. The black hole singularity at $uv = 1$, rather than being the future as appearing to an observer in regions I,II, occurs at the end of the spatial world to an observer in region V. (Region III has exactly the "time reversed" properties of region II, and is referred to as a white hole. Region IV has properties identical to region I, and region VI is a continuation over the past singularity. An observer in region I cannot communicate with any observer in region IV.)

We now come to a striking property of the stringy black hole: we will argue that *target space duality interchanges the horizon with the singularity* [137, 83]. Therefore, signals can pass smoothly through the singularity $uv = 1$. Such signals should be associated with the winding modes of the string as was discussed in the Euclidean case. To understand this point let us present the duality transformation of the original Lorentzian background.

The dual Lorentzian black hole is given by the analytic continuation of the dual Euclidean black hole. The metric background is thus described by the line element

$$(ds')^2 = dr^2 - \epsilon^2 \coth^2 \epsilon r \; dt^2. \tag{4.5.35}$$

We will repeat a procedure similar to the coordinate transformation done for the original black hole. With u, v coordinates defined by

$$v = \cosh \epsilon r \; e^{\epsilon^2 t}, \qquad u = \cosh \epsilon r \; e^{-\epsilon^2 t}, \tag{4.5.36}$$

one gets

$$(ds')^2 = -\frac{1}{\epsilon^2} \frac{dudv}{1 - uv}. \tag{4.5.37}$$

The line element $(ds')^2$ of the dual black hole is thus identical to the line element (4.5.33) of the original black hole. In the u, v coordinates one cannot distinguish between the original background and its duality partner. This is not bad since the two theories must be equivalent as conformal field theories. What is therefore new in regarding the dual black hole? The answer is remarkable: duality interchanges the asymptotically flat space-time, corresponding to the Lorentzian cigar, with the space-time inside the black hole singularity; that is, regions I and V in figure 4.A are interchanged. (Region II is transformed to itself; regions IV and VI are interchanged and region III is transformed to itself). As a consequence, the physics studied by an observer in region I is equivalent to the physics in region V. In particular, the dual transform of a signal leaving region I will be a signal leaving region V. As it was discussed before, this is

related to the quantum symmetry which interchanges momentum modes with winding modes, a result of target space duality.

So far we have described the Lorentzian $SL(2,\mathbf{R})/U(1)$ black hole and its target space duality partner in the graviton and dilaton background. To conclude the study of this example, let us now consider the propagation of a small "tachyon" disturbance in the Lorentzian black hole [31], and verify explicitly its invariance under duality. In the black hole space the tachyon effective action to lowest order in α' is

$$L(T) = \int dudv[(1-uv)\partial_u T\partial_v T - 8\epsilon^2 T^2]. \tag{4.5.38}$$

The tachyon field equation is therefore

$$\partial_u[(1-uv)\partial_v T] + \partial_v[(1-uv)\partial_u T] + 16\epsilon^2 T = 0. \tag{4.5.39}$$

Define

$$z = 1 - uv, \tag{4.5.40}$$

then, with the ansatz $T = T(z)T(t)$, the equation for $T(z)$ is a hypergeometric differential equation

$$z(1-z)\frac{\partial^2}{\partial z^2}T(z) + (1-2z)\frac{\partial}{\partial z}T(z) + 8\epsilon^2 T(z) = 0. \tag{4.5.41}$$

The differential equation is invariant under the duality transformation $z \to 1-z$. There are two linearly independent solutions to eq. (4.5.41) given by

$$T_1(z) = F(\alpha, 1-\alpha; 1; z), \qquad \alpha = -8\epsilon^2 + \mathcal{O}(\epsilon^4), \tag{4.5.42}$$

$$T_2(z) = T_1(1-z). \tag{4.5.43}$$

Here $F(\alpha,\beta;\gamma;z)$ is the hypergeometric function. The solution T_2 is dual to the solution T_1. The function $T_2(z)$ has a singularity at $z=0$ $(uv=1)$, although regular at $z=1$ $(uv=0)$, and as a result, a physical singularity is created at $uv=1$. The dual function $T_1(z)$ has a singularity at $z=1$ $(uv=0)$, although regular at $z=0$ $(uv=1)$.

To summarize, the target space duality transformation of the $SL(2,\mathbf{R})/U(1)$ space-time leads to strange stringy properties of the black hole. Although we have discussed a particular black hole in string theory, these properties shall be generalized to other black hole solutions (as well as cosmological solutions, monopoles, dipoles and other topological backgrounds). There is an extensive discussion on the generalization of the $D=2$ black hole duality to other backgrounds in the literature (for example, see [18, 271, 273, 62, 24, 147, 68, 63, 170, 172, 240, 171, 135, 191] [142, 185, 174, 125, 198].) In the next section we will describe some more applications of duality.

[31] The propagation of a tachyon disturbance in the Euclidean black hole was studied in [92].

4.6 Applications II: Charged Black Objects in D=2,3

In this section we explore more consequences of the discrete duality symmetries [147]. We first discuss an exact $D = 3$ closed string background that is independent of $d = 2$ coordinates [179]. We then turn to $D = 2$ heterotic backgrounds [71, 224]. In both cases, we find that uncharged black compact objects (strings or holes) are equivalent to charged $D = 2$ black holes [173, 147].

4.6.1 The Closed String Example

The simplest nontrivial example after the $D = 2$ black hole duality, presented in section 4.5, is a compact black string obtained by attaching a circle to every point of the $D = 2$ black hole space-time $(SL(2,\mathbf{R})_k/U(1) \times U(1))$. To leading order, the action is

$$S_{BlackString} = \frac{1}{2\pi}\int d^2z \left[k(\partial x\bar{\partial}x + \tanh^2 x\partial\theta^1\bar{\partial}\theta^1) + \alpha\partial\theta^2\bar{\partial}\theta^2 - \frac{1}{4}\phi(x)R^{(2)}\right], \tag{4.6.1}$$

where

$$\phi(x) = \phi_0 + \log(\cosh^2 x)\,. \tag{4.6.2}$$

The first term in $S_{BlackString}$ is a σ-model in the Euclidean black hole metric, and the second term corresponds to a free scalar field compactified on a circle of radius $\sqrt{\alpha}$. This $D = 3$ background is independent of $d = 2$ coordinates θ^i, and is described by the background matrix (4.2.8)

$$\mathcal{E} = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}, \tag{4.6.3}$$

where

$$E = \begin{pmatrix} k\tanh^2 x & 0 \\ 0 & \alpha \end{pmatrix}, \qquad F = k\,. \tag{4.6.4}$$

The group of generalized duality transformations $O(2,2,\mathbf{Z})$ maps this background into other backgrounds that (in general) have different space-time interpretations. Since $F_1 = F_2 = 0$ in (4.6.3, *cf.* 4.2.8), $O(2,2,\mathbf{R})$ acts on the background by transforming only E and ϕ as given in (4.2.31) and (4.2.37). A particularly interesting point on the orbit of $O(2,2,\mathbf{Z})$ is reached by acting with the element

$$g = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & -\Theta \\ 0 & I \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}, \tag{4.6.5}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.6.6}$$

One finds that E and ϕ are transformed to

$$E' = g(E) = \frac{1}{1+\alpha k \tanh^2 x}\begin{pmatrix} k\tanh^2 x & \alpha k \tanh^2 x \\ -\alpha k \tanh^2 x & \alpha \end{pmatrix} \tag{4.6.7}$$

$$= \frac{1}{\cosh^2 x - \lambda}\begin{pmatrix} k(1-\lambda)\sinh^2 x & \lambda \sinh^2 x \\ -\lambda \sinh^2 x & \lambda \cosh^2 x/k \end{pmatrix}$$

$$\phi'(x) = \phi_0 - \log(1-\lambda) + \log(\cosh^2 x - \lambda)\,, \tag{4.6.8}$$

where

$$\lambda = \frac{k\alpha}{1+k\alpha}\,. \tag{4.6.9}$$

This gives the worldsheet action

$$S_{charged} = \frac{1}{2\pi}\int d^2z \Big[k\left(\partial x\bar\partial x + \frac{(1-\lambda)\sinh^2 x}{\cosh^2 x - \lambda}\partial\theta^1\bar\partial\theta^1\right) \tag{4.6.10}$$

$$+ \frac{\lambda\sinh^2 x}{\cosh^2 x-\lambda}(\partial\theta^1\bar\partial\theta^2 - \partial\theta^2\bar\partial\theta^1) + \frac{\lambda\cosh^2 x}{k(\cosh^2 x-\lambda)}\partial\theta^2\bar\partial\theta^2 - \frac{1}{4}\phi'(x)R^{(2)}\Big]\,,$$

and, after Wick-rotating $\theta^1 \to it$, it corresponds to a charged black hole of the type found in [179] with mass

$$M = (1-\lambda)M_0 = \sqrt{\frac{2}{k}}e^{\phi_0}\,, \qquad M_0 = \sqrt{\frac{2}{k}}e^{\phi_0'}\,, \tag{4.6.11}$$

and charge

$$\mathcal{Q} = \sqrt{\lambda(1-\lambda)}\frac{2M_0}{k} = \sqrt{\frac{\lambda}{1-\lambda}}\frac{2M}{k}\,, \tag{4.6.12}$$

where $\phi_0' = \phi_0 - \log(1-\lambda)$ is the constant part of the dual dilaton (4.6.8). The action (4.6.10) is related to the σ-model action for the coset $(SL(2,\mathbf{R})_k\times U(1))/U(1)$ [179] by rescaling $\theta^2 \to k\theta^2$.[32]

This shows that *the charged black hole is equivalent to the compact black string* as a conformal field theory. We now consider some particular limits of this solution.

The limit $\alpha \to 0$ in (4.6.1) corresponds to the direct product of the $2D$ black hole and a circle with radius $r \to 0$, and the limit $\alpha \to \infty$ is related to the previous one by the $R \to 1/R$ circle duality. These limits correspond to the limits $\lambda \to 0, 1$ in (4.6.9). In $S_{charged}$ (4.6.10), the background at $\lambda \to 0$ is the direct product of the $2D$ black hole background and the same $r \to 0$ circle; however, the $\lambda \to 1$ limit gives the action (modulo an integer total derivative term)

$$S_{\lambda\to 1} = \frac{1}{2\pi}\int d^2z\left[k\partial x\bar\partial x + \frac{1}{k}\coth^2 x\partial\theta^2\bar\partial\theta^2 + (1-\lambda)\partial\theta^1\bar\partial\theta^1 - \frac{1}{4}\phi'(x)R^{(2)}\right], \tag{4.6.13}$$

$$\phi'(x) = \phi_0 - \log(1-\lambda) + \log(\sinh^2 x)\,.$$

[32] Our k matches [285], which is $2k$ of [179].

The background in this action is the direct product of the dual $2D$ black hole background and a $r = 0$ limit of a circle. In both cases, the degenerate limits are equivalent as CFTs to a *noncompact* black string.

4.6.2 The Heterotic String Example

Following [147], we focus on the example of [224], which is a $D = 2$ heterotic string, with internal degrees of freedom taking values in a standard 12-dimensional lattice (the vector weights of $SO(24)$)[33]. We find that *a family of charged black holes (and naked singularities) is dual to a neutral one*, which is the exact CFT given by the heterotic $D = 2$ black hole [224].

We start with a heterotic $D = 2$ action:

$$S_{Heterotic} = \frac{1}{2\pi}\int d^2z \left[k(\partial x\bar{\partial}x + \tanh^2 x\partial\theta\bar{\partial}\theta) + \partial Y^A\bar{\partial}Y^A - \frac{1}{4}\phi(x)R^{(2)} + (\text{fermionic terms})\right], \tag{4.6.14}$$

where $A = 1, \ldots, 12$, $k = 5/2$ (for criticality), and $\phi = \phi_0 + \log(\cosh^2 x)$. This action describes a neutral heterotic $D = 2$ black hole. The conformal field theory (4.6.14) corresponds to a background (4.4.3)

$$\Xi = \begin{pmatrix} k & 0 & 0 \\ 0 & k\tanh^2 x & \\ 0 & 0 & I \end{pmatrix}, \tag{4.6.15}$$

where the internal background I is the 12×12 identity matrix corresponding to the vector weights of $SO(24)$. Only a 2×2 block E of the matrix Ξ is affected by the discrete transformations we discuss in this example:

$$E = \begin{pmatrix} k\tanh^2 x & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.6.16}$$

By transforming E and ϕ with a group element $g_n \in O(1, 2, \mathbf{Z}) \subset O(1, 13, \mathbf{Z})$ (where n is an arbitrary integer):

$$g_n = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & n\Theta \\ 0 & I \end{pmatrix}\begin{pmatrix} A_n^t & 0 \\ 0 & A_n^{-1} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -n^2 & n \\ -n & 0 \end{pmatrix} \end{pmatrix}, \tag{4.6.17}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, \tag{4.6.18}$$

one finds

$$g_n(E) = E'_n = \begin{pmatrix} (n^2 + k\tanh^2 x)^{-1} & -2n(n^2 + k\tanh^2 x)^{-1} \\ 0 & 1 \end{pmatrix}, \tag{4.6.19}$$

[33] More precisely, only space-time bosons have internal quantum numbers in the vector representations of $SO(24)$; space-time fermions have internal quantum numbers in the spinor representations of $SO(24)$.

$$\phi'(x) = \phi_0 + \log(n^2+k) + \log(\cosh^2 x - \frac{k}{n^2+k})\,. \tag{4.6.20}$$

After rescaling

$$\theta \to \frac{k+n^2}{\sqrt{k}} t\,, \tag{4.6.21}$$

and defining the coordinate r to be a linear function of the dilaton ϕ' (4.6.20), given by

$$r = \frac{1}{Q}\log(\cosh^2 x - \frac{k}{n^2+k})\,, \tag{4.6.22}$$

where Q is a constant determined below, the background (4.6.19) gives rise to the action[34]

$$\begin{aligned} S_{charged} = \frac{1}{2\pi}\int d^2z \Big[& f(r)\partial t\bar\partial t + f(r)^{-1}\partial r\bar\partial r - A(r)\partial t\bar\partial Y^1 + \partial Y^A\bar\partial Y^A \\ & -\frac{1}{4}\phi'(r)R^{(2)} + (\text{fermionic terms})\Big]\,. \end{aligned} \tag{4.6.23}$$

Here

$$f(r) = 1 - 2me^{-Qr} - q^2 e^{-2Qr}, \tag{4.6.24}$$

$$A(r) = nQ + 2qe^{-Qr}\,, \qquad \phi'(r) = Qr + \phi_0 + \log(n^2+k)\,,$$

where $Q = 2/\sqrt{k'}$ is determined by the normalization of $\mathcal{G}_{rr}$ in (4.6.23), and

$$2m = \frac{n^2-k'}{n^2+k'}\,, \qquad q = \frac{n\sqrt{k'}}{n^2+k'}\,. \tag{4.6.25}$$

Following [285] we have replaced k with $k' = k-2 = 1/2$ in (4.6.25). By Wick-rotating $t \to it$, along with $q \to -iq$ (necessary to maintain hermiticity of the action), the theory (4.6.23) with $|n| > 1$ describes a $D=2$ charged black hole with mass and charge [224]

$$M = Q(n^2-k')e^{\phi_0}\,, \qquad \mathcal{Q} = n\sqrt{8}e^{\phi_0}\,. \tag{4.6.26}$$

For $n = -1, 0, 1$, the theory (4.6.23) describes a naked singularity.

We emphasize that these backgrounds, for all n, are different space-time interpretations of the same CFT: the CFT given by the neutral heterotic $D=2$ black hole.

[34] Recall that $\mathcal{G}_{tt} = \Xi'_{tt} - \frac{1}{4}\mathcal{A}^2$, see eq. (4.4.3).

4.7 Applications III: Cosmological Backgrounds with Abelian Symmetries

In this section we discuss a class of cosmological solutions in string theory (in the presence of Maxwell fields) that are obtained by $O(d,d,\mathbf{R})$ transformations of simple backgrounds with d Abelian symmetries, following [142]. Similar cosmological solutions are discussed extensively in the literature (for examples, see [231, 202, 235, 123, 192, 2, 193]). In some of the examples we will find a (closed) expanding universe. In all the cases for which the universe has a smooth and complete initial value hypersurface, a naked singularity can form only at the time when the universe collapses. The discrete symmetry group $O(d,d,\mathbf{Z})$ identifies different cosmological solutions with a background corresponding to a (relatively) simple conformal field theory (CFT), and therefore, may be useful in understanding the properties of naked singularities in string theory. The consequences of duality in cosmological string backgrounds were also discussed, for instance, in [32, 160, 271, 273]

To find realistic cosmological solutions in string theory, we shall represent a classical solution by $M \times K$, where M is a $2d$ CFT with a four dimensional target space-time, and with a central charge $c = 4$, and K is some internal space represented also by a CFT. Moreover, although the formation of singularities is one of the interesting questions in such solutions, we do not want the singularities to be "built in" the initial conditions. Namely, we want M to have a smooth and complete initial value hypersurface.

We shall first construct new solutions by $O(d,d,\mathbf{R})$ rotations with respect to Abelian symmetries of the space-time background M. Let us consider the 4-D line element

$$d\mathbf{s}^2 = -dt^2 + ds^2 + g(t)^2 d\theta_1^2 + f(s)^2 d\theta_2^2. \tag{4.7.1}$$

We want this background to correspond to a CFT[35]. There are four nontrivial possibilities for $g(t)$ and four possibilities for $f(s)$. To leading order in α' the possibilities are[36]:

$$g(t) = \tan(t), \cot(t), \tanh(t) \ \text{ or } \ \coth(t),$$

and

$$f(s) = \tan(s), \cot(s), \tanh(s) \ \text{ or } \ \coth(s).$$

The dilaton is then given by

$$\phi(s,t) = \phi_0 + \log(\bar{f}(s)^2 \bar{g}(t)^2), \tag{4.7.2}$$

where ϕ_0 is a constant and

$$\bar{g}(t) = \cos(t), \sin(t), \cosh(t) \ \text{ or } \ \sinh(t),$$

[35] More precisely, we should start with $d\mathbf{s}^2 = k(-dt^2 + g(t)^2 d\theta_1^2) + k'(ds^2 + f(s)^2 d\theta_2^2)$, and choose k and k' such that $c = 4$. To leading order in α' this condition is $k = k'$, and for simplicity we take $k = k' = 1$.

[36] All these possibilities correspond to the exact CFT given by a direct product of two cosets $\frac{G}{H}$, where G is either $SU(2,\mathbf{R})$ or $SL(2,\mathbf{R})$ and H is either $U(1)$ or $\mathbf{R}$ (see section 4.5).

$$\bar{f}(s) = \cos(s), \sin(s), \cosh(s) \ \text{ or } \ \sinh(s),$$

respectively.

The background in (4.7.1),(4.7.2) is a $D = 4$ curved background that is independent of $d = 2$ coordinates. It has one time-like coordinate t, and three space-like coordinates s, θ_1, θ_2. The background is time dependent, and therefore, we say that it describes a cosmological solution to Einstein's equations. The 2×2 matrix $E_{ij}(s,t)$ (4.2.8) in the worldsheet action (4.2.7) (with $i, j = 1, 2$ and $a, b = s, t$) is

$$E(s,t) = \begin{pmatrix} g(t)^2 & 0 \\ 0 & f(s)^2 \end{pmatrix}. \tag{4.7.3}$$

We can now generate new cosmological solutions by acting on (E, ϕ) in (4.7.3) and (4.7.2) with $O(2,2,\mathbf{R})$ transformations as described in section 4.2. Let us discuss a particular one parameter sub-family of $O(2,2,\mathbf{R})$ rotations. By transforming E and ϕ with the group element $g_b \in O(2,2,\mathbf{R})$ (where b is an arbitrary real number):

$$g_b = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & b\Theta \\ 0 & I \end{pmatrix}, \tag{4.7.4}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.7.5}$$

one finds that the new background matrix, E'_b, is

$$g_b(E) = E'_b = (E + b\Theta)^{-1} = \frac{1}{f(s)^2 g(t)^2 + b^2} \begin{pmatrix} f(s)^2 & -b \\ b & g(t)^2 \end{pmatrix}. \tag{4.7.6}$$

Namely, the new background matrix is given by adding a constant antisymmetric background $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ to E, and then using duality to invert the background matrix. From eq. (4.2.23) and the discussion in subsection 4.2.4 one finds that the new dilaton is

$$\phi'(s,t) = \phi_0 + \log\left[\bar{f}(s)^2 \bar{g}(t)^2 \left(f(s)^2 g(t)^2 + b^2\right)\right] \ . \tag{4.7.7}$$

For the particular choice

$$\begin{aligned} f(s) &= \tan(s) \quad , \qquad g(t) = \cot(t) \ , \\ \bar{f}(s) &= \cos(s) \quad , \qquad \bar{g}(t) = \sin(t) \ , \end{aligned} \tag{4.7.8}$$

the conformal background corresponds to a closed expanding universe [235] [37].

Let us consider now the case in which the internal space K also has Abelian symmetries. Then by $O(d,d,\mathbf{R})$ rotations one can turn on gauge fields, namely, non-trivial elements of the

[37] To get the solution in [235] (corresponding to the σ-model of the $c = 4$ coset $\frac{SL(2,\mathbf{R})\times SU(2,\mathbf{R})}{\mathbf{R}\times\mathbf{R}}$) one should rescale $\theta_1 \to b\theta_1$, reintroduce k and k', and take $k = k'$ very large such that the maximal size of the universe is of order k, and the central charge of M is $c = 4$ (to leading order in $1/k$).

background matrix interpolating between the space-time (M) indices and the internal space (K) indices.

The action (4.2.7) can be generalized adding internal degrees of freedom, *i.e.* extra compactified coordinates Y^A, $A = 1, ..., d_{int}$. (The coordinates Y^A can be regarded as either the internal coordinates of a bosonic string or the extra coordinates of a heterotic string, as in section 4.4 [147]). We start from the action (4.4.1), and we choose the initial structure of the background as before, i.e., the block E_{ij} is again given by eq. (4.7.3), $F^1 = F^2 = 0$, and $F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and we start with a vanishing background gauge field: $\mathcal{A} = 0$ in (4.4.1).

It is now possible to make a more general rotation using the group $O(2, 2+d_{int}, \mathbf{R}) \subset O(2+d_{int}, 2+d_{int}, \mathbf{R})$ acting by the fractional linear transformations (4.2.31) on the $(2+d_{int})\times(2+d_{int})$ matrix (see eq. (4.4.3))

$$\Xi = \begin{pmatrix} E_{ij} & 0 \\ 0 & E_{AB} \end{pmatrix}, \tag{4.7.9}$$

such that

$$\Xi' = \begin{pmatrix} E'_{ij} + \frac{1}{4}\mathcal{A}_{iA}(G^{-1})^{AB}\mathcal{A}_{jB} & \mathcal{A}_{iA} \\ 0 & E_{AB} \end{pmatrix}. \tag{4.7.10}$$

Here G_{AB} is defined as in eq. (4.4.2). The structure of Ξ' is such that $E'_{ij} = G'_{ij} + B'_{ij}$ gives the correct metric after a dimensional reduction from $4 + d_{int}$ dimensions to four space-time dimensions. The structure of the second line in the matrix Ξ' allows the internal coordinates to be the extra coordinates of a heterotic string as explained in sections 2.5 and 4.4.

The new background in (4.7.10) corresponds to the heterotic worldsheet action given in (4.4.1) with non-vanishing background gauge field $\mathcal{A}$, or to a bosonic action given by (4.4.1) (neglecting worldsheet fermions in the bosonic case). Both the old background and the new metric, antisymmetric tensor, dilaton and gauge fields are solutions of the equations of motion derived from the effective action

$$S_{eff} = \int d^4X\sqrt{-G}e^{\phi}\left[R^{(4)} + (\nabla\phi)^2 - \frac{1}{12}H^2 - \frac{1}{16}\mathrm{Tr}F^2\right] . \tag{4.7.11}$$

(The old background has $H_{\mu\nu\rho} = F_{\mu\nu} = 0$.)

For simplicity, we now consider the case in which $d_{int} = 1$, *i.e.* we start from a five dimensional background where the fifth coordinate is compact and by $O(2, 3, \mathbf{R}) \subset O(3, 3, \mathbf{R})$ rotations we introduce a Maxwell field $\mathcal{A}_i$. (The discussion can be easily generalized to the internal dimension d_{int} that is needed for criticality.)

We fix $E_{AB} = 1$ and we rotate Ξ by the $g_{a,b,c} \in O(2, 3, \mathbf{R}) \subset O(3, 3, \mathbf{R})$ matrix

$$g_{a,b,c} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \tag{4.7.12}$$

where

$$\Theta = \begin{pmatrix} 0 & b & c \\ -b & 0 & a \\ -c & -a & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.7.13}$$

I is the 3×3 identity matrix and a, b, c are arbitrary real numbers.

After the rotation, the non-zero components of the metric G'_{IJ}, antisymmetric tensor B'_{IJ}, gauge field $\mathcal{A}_I$ and dilaton ϕ', are:

$$\begin{aligned}
&G'_{tt} = -1 && G'_{ss} = 1 \\
&G'_{\theta_1\theta_1} = \frac{1}{\Delta^2}\left[g(t)^2\left(a^2 + f(s)^2\right)^2 + f(s)^2\,(ac+b)^2\right] \\
&G'_{\theta_1\theta_2} = \frac{1}{\Delta^2}\left[b\left(a^2 g(t)^2 - c^2 f(s)^2\right) - ac\left(a^2 g(t)^2 + c^2 f(s)^2 + 2f(s)^2 g(t)^2\right)\right] \\
&G'_{\theta_2\theta_2} = \frac{1}{\Delta^2}\left[f(s)^2\left(c^2 + g(t)^2\right)^2 + g(t)^2\,(ac-b)^2\right] \\
&\mathcal{A}_{\theta_1} = \frac{1}{2\Delta}\left[ab - cf(s)^2\right] && \mathcal{A}_{\theta_2} = \frac{1}{2\Delta}\left[-bc - ag(t)^2\right] \\
&B'_{\theta_1\theta_2} = -\frac{b}{\Delta} && \phi' = \phi_0 + \log\left[\bar{f}(s)^2 \bar{g}(t)^2 \Delta\right] \\
&\det\left[-G'\right] = \frac{g(t)^2 f(s)^2}{\Delta^2},
\end{aligned} \tag{4.7.14}$$

where

$$\Delta = b^2 + a^2 g(t)^2 + c^2 f(s)^2 + g(t)^2 f(s)^2 \ . \tag{4.7.15}$$

Obviously, for $a = c = 0$, one gets back the space-time solutions in (4.7.6) and (4.7.7).

It is also possible to make a Weyl rescaling in order to bring the effective action (4.7.11) to the Einstein form

$$S^E_{eff} = \int d^4X \sqrt{-G^E}\left[R^{E\,(4)} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{12}e^{2\phi}H^2 - \frac{1}{16}e^{\phi}\mathrm{Tr}F^2\right], \tag{4.7.16}$$

where the Einstein metric $G^E_{\mu\nu} = e^{\phi}G_{\mu\nu}$ is used to contract the indices.

Notice that when f and g are given by eq. (4.7.8) one has

$$\begin{aligned}
\phi' \;=\; \phi_0 + \log\Big[&b^2\sin^2(t)\cos^2(s) + a^2\cos^2(t)\cos^2(s) \\
&+ c^2\sin^2(t)\sin^2(s) + \cos^2(t)\sin^2(s)\Big] \ .
\end{aligned} \tag{4.7.17}$$

When the parameters a, b and c are non-zero, the dilaton is finite for any value of the space-time coordinates s, t (and therefore, the Weyl rescaling to the Einstein form does not change the properties of the universe described above). When some of the parameters a, b, c are zero, the dilaton diverges in particular points in space-time. Explicitly, if $a = 0$ the dilaton diverges at

$t = 0$, $s = 0$; if $c = 0$ it diverges at $t = \pi/2$, $s = \pi/2$; and if $b = 0$ it diverges at $t = \pi/2$, $s = 0$ ($t, s \in [0, \pi/2]$). The divergences of the dilaton appear when a curvature singularity is formed.

Let us discuss briefly the target space properties of some of the models presented above. From eq. (4.7.14), we see that the volume of the universe, described by that class of solutions, approaches zero when $t \to 0$ (this is because $g(t = 0) = 0$ or ∞ for all four possible choices of g). Therefore, the universe starts from a "big bang." Later, the universe can either be an expanding and contracting closed universe, in some cases collapsing at the end, or an open universe. The different types of universes depend on the different choices of the functions f, g and the parameters a, b, c. (We do not consider the models with singularities that are built in the initial conditions and remain for any time t, as it happens for instance when $a = b = 0$, $c \neq 0$, and for functions f and g given by eq. (4.7.8), or in the example of [202].)

Consider again the case when the functions f and g are given by eq. (4.7.8) (with the parameters b and c not both zero). For this example, the backgrounds in (4.7.14) describe an anisotropic expanding and contracting universe, in the presence of an antisymmetric tensor background, a Maxwell field, and a dilaton. The topology of a spatial slice is S^3, the three sphere, and the universe recollapses at the time $t = \frac{\pi}{2}$.

Next we discuss the formation of singularities in the universe described above. There are different cases depending on the parameters a, b, and c. For $a = 0$ the universe is singular at the "big bang," namely, when $t = 0$ a singularity exists at $s = 0$. For $b = 0$ ($c = 0$), a naked singularity is about to form at $s = 0$ ($s = \pi/2$) at the time $t = \pi/2$, when the universe recollapses. For non-vanishing parameters a, b and c there are no singularities. Therefore, a naked singularity may form only at the times when the universe collapses.

Because there is no singularity for non-vanishing parameters a, b and c, there are indications that one can *remove target space singularities in string theory* by continuous $O(d, d, \mathbf{R})$ rotations [142, 123].

Finally, we remark briefly on some consequences of the duality group symmetries. In the examples we have discussed, the matrix g_b in eq. (4.7.4) ($g_{a,b,c}$ in eq. (4.7.12)) is an element of the duality group $O(2, 2, \mathbf{Z})$ ($O(3, 3, \mathbf{Z})$) if $b \in \mathbf{Z}$ ($a, b, c \in \mathbf{Z}$). In particular, all the solutions (4.7.6) with $b \in \mathbf{Z}$ are equivalent as CFTs to the $b = 0$ case, for which the background is a direct product of (a suitable analytic continuation) of two $D = 2$ black hole solutions discussed in section 4.5. Moreover, all the solutions (4.7.14) with $a, b, c \in \mathbf{Z}$ are also equivalent CFTs. Therefore, these closed expanding universes, in the presence of electromagnetic currents, are also equivalent as CFTs to the direct product of (a suitable analytic continuation) of two $D = 2$ black hole solutions, and an extra coordinate compactified on a circle.

4.8 Axial-Vector Duality as a Gauge Symmetry

The interpretation of discrete symmetries in target space as gauge symmetries was already discussed in the flat case in section 2.6. Can we also interpret the target space dualities in *curved backgrounds* as spontaneously broken gauge symmetries (of an underlying gauge algebra) in string theory? In this section we answer in the affirmative, concerning particular elements of the $O(d,d,\mathbf{Z})$ duality group, following [139]. Namely, we show that any element of $O(d,d,\mathbf{Z})$, relating an axially gauged Abelian quotient of a WZNW model to its vector gauging [196], is an element of a spontaneously broken gauge group, in the sense discussed in section 2.6. In particular, this proves that the duality, relating the σ-model background of an axially gauged Abelian coset to a vectorially gauged coset, is an exact symmetry in string theory. This is true for compact groups, as well as for non-compact groups, and therefore, it has important implications for the black-hole duality presented in section 4.5, and for the study of singularities in string theory discussed in section 4.7. The extension to the full $O(d,d,\mathbf{Z})$ might be done along the lines of section 2.6 [141].

This section is organized as follows: In subsection 4.8.1 we start with the simplest non-trivial case, the $SU(2)$ (or $SL(2,\mathbf{R})$) model and its one-parameter sub-class of marginal deformations. In subsection 4.8.2 we extend the discussion to a general group. In both cases, we discuss the action of duality on the line of marginal deformations, and its relation to a broken gauge transformation. It will turn out that the two boundaries of the deformation line (which are related by this duality) correspond to the axial coset and the vector coset (times a non-compact scalar field).

4.8.1 $J\bar{J}$ Deformation of $SU(2)$ or $SL(2,\mathbf{R})$ WZNW Models and Duality as a Gauge Symmetry

In this subsection we consider duality as a spontaneously broken gauge symmetry for the simplest nontrivial case, namely, duality acting on the deformation line of $SU(2)$ or $SL(2,\mathbf{R})$ WZNW models[38].

The worldsheet action of the $SU(2)_k$ WZNW model is given (in Euler angle parametrization as in (4.5.7)) by

$$\begin{aligned} S[x,\theta_1,\theta_2] &= S_1 + S_a + S[x], \\ S_1 &= \frac{k}{2\pi}\int d^2z(\partial\theta_1\bar{\partial}\theta_1 + \partial\theta_2\bar{\partial}\theta_2 + 2\Sigma(x)\partial\theta_2\bar{\partial}\theta_1), \\ S_a &= \frac{k}{2\pi}\int d^2z(\partial\theta_2\bar{\partial}\theta_1 - \partial\theta_1\bar{\partial}\theta_2), \end{aligned}$$

[38] We define the conformal field theory corresponding to $SL(2,\mathbf{R})$ to be the one regularized by its Euclidean continuation. This regularization has been shown to provide a consistent path integral prescription for the $SL(2,\mathbf{R})$ model [127].

$$S[x] \;=\; \frac{k}{2\pi}\int d^2z\;\partial x\bar{\partial}x - \frac{1}{8\pi}\int d^2z\;\phi_0 R^{(2)}, \tag{4.8.1}$$

where $\Sigma(x)=\cos 2x$, and ϕ_0 is a constant dilaton. The action of the $SL(2,\mathbf{R})_k$ WZNW model in (4.5.8) is obtained from (4.8.1) by taking $x\to ix$ and $k\to -k$.

As discussed in subsection 4.2.1, the antisymmetric term S_a in (4.8.1) is (locally) a total derivative, and therefore, may give only topological contributions, depending on the periodicity of the coordinates θ. To specify the periodicity, we follow subsection 4.2.1 and define

$$\theta=\theta_2-\theta_1\;,\qquad \tilde{\theta}=\theta_1+\theta_2\;, \tag{4.8.2}$$

such that

$$\theta\equiv\theta+2\pi\;,\qquad \tilde{\theta}\equiv\tilde{\theta}+2\pi\;. \tag{4.8.3}$$

In these coordinates the action becomes

$$S[x,\theta,\tilde{\theta}]=\frac{1}{2\pi}\int d^2z(\partial\theta,\partial\tilde{\theta},\partial x)\begin{pmatrix} E & 0\\ 0 & k\end{pmatrix}\begin{pmatrix}\bar{\partial}\theta\\ \bar{\partial}\tilde{\theta}\\ \bar{\partial}x\end{pmatrix}-\frac{1}{8\pi}\int d^2z\phi_0 R^{(2)}. \tag{4.8.4}$$

Here we have organized the background into a 3×3 block diagonal matrix, where the 2×2 matrix block E is

$$E=\frac{k}{2}\begin{pmatrix}1-\Sigma & 1+\Sigma\\ -(1+\Sigma) & 1+\Sigma\end{pmatrix}. \tag{4.8.5}$$

The action S in eq. (4.8.4) is manifestly invariant under the $U(1)_L\times U(1)_R$ affine symmetry generated by the chiral current J and anti-chiral current $\bar{J}$ given by

$$\begin{aligned} J&=\frac{k}{2}[-(1-\Sigma)\partial\theta+(1+\Sigma)\partial\tilde{\theta}]\;,\\ \bar{J}&=\frac{k}{2}[(1-\Sigma)\bar{\partial}\theta+(1+\Sigma)\bar{\partial}\tilde{\theta}]\;. \end{aligned} \tag{4.8.6}$$

In addition, there are two extra chiral currents, and two extra anti-chiral currents, completing the affine $SU(2)_L\times SU(2)_R$ (or $SL(2,\mathbf{R})_L\times SL(2,\mathbf{R})_R$) symmetry of the WZNW model.

We can deform the action S to new conformal backgrounds by adding to it any truly marginal deformation. We will focus on marginal deformations that are obtained as a linear combination of chiral currents times a linear combination of anti-chiral currents. It is important to recall that all the deformations that are equivalent under the action of the symmetry group give rise to equivalent CFTs (although not necessarily to backgrounds that are related by coordinate transformations). In the following we deform the WZNW action with the $J\bar{J}$ marginal operator, as was done in refs. [167, 139].

Deforming with $J\bar{J}$, the affine symmetry is broken to $U(1)_L\times U(1)_R$. The $U(1)$ chiral and anti-chiral currents of the deformed theory can be found and will be presented below. Once

choosing a $J\bar{J}$ deformation of the WZNW point (4.8.1) for specific J and $\bar{J}$, the deformed theories are described in terms of a one parameter family of $O(2,2,\mathbf{R})$ rotations.

Let us describe these rotations. The backgrounds corresponding to the $J\bar{J}$ deformation of the WZNW point, with J and $\bar{J}$ given in (4.8.6), are constructed by an $O(2,2,\mathbf{R})$ transformation (4.2.31) of the WZNW background (4.8.5) by the element [167, 139]

$$g_\alpha = \begin{pmatrix} I & \cos^2\alpha(k-\tan\alpha)\epsilon \\ 0 & I \end{pmatrix} \begin{pmatrix} A(\alpha) & 0 \\ 0 & (A(\alpha)^t)^{-1} \end{pmatrix} \begin{pmatrix} C(\alpha) & S(\alpha) \\ S(\alpha) & C(\alpha) \end{pmatrix} \begin{pmatrix} I & -k\epsilon \\ 0 & I \end{pmatrix}, \quad (4.8.7)$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A(\alpha) = \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\alpha(1+k\tan\alpha) \end{pmatrix}$$
$$C(\alpha) = \cos\alpha\, I, \qquad S(\alpha) = \sin\alpha\, \epsilon. \quad (4.8.8)$$

Namely,

$$\frac{1}{k} g_\alpha(E) \equiv E_{R(\alpha)} = \frac{1}{1+R^2\frac{1-\Sigma}{1+\Sigma}} \begin{pmatrix} \frac{1-\Sigma}{1+\Sigma} & 1 \\ -1 & R^2 \end{pmatrix}, \quad (4.8.9)$$

where

$$R(\alpha)^2 = (1+k\tan\alpha)^2. \quad (4.8.10)$$

The parameter R can be interpreted, geometrically, as the radius of the Cartan torus.

The dilaton field ϕ also transforms under $O(2,2,\mathbf{R})$ rotations, as discussed in sections 4.2, 4.3. At the WZNW point the dilaton is the constant $\phi = \phi_0$ appearing in (4.8.1). At the point R the dilaton is (4.2.23)

$$\phi(R) = \phi_0 + \frac{1}{2}\log\left(\frac{\det G(1)}{\det G(R)}\right), \quad (4.8.11)$$

where $G(R)$ is the symmetric part of E_R.

Although we have generated the R-dependent backgrounds by $O(2,2,\mathbf{R})$ rotations, which are correct only to leading order in α', it can be shown [139] that there is a scheme in which these backgrounds are exact to all orders in α', for any R.

Two special backgrounds occur at the boundaries of the R-modulus space. At $R = 0$ ($\alpha = \tan^{-1}(-\frac{1}{k})$) the background matrix is

$$E_{R=0} = \begin{pmatrix} \frac{1-\Sigma}{1+\Sigma} & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1-\Sigma}{1+\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \mod \Theta - \text{shift}. \quad (4.8.12)$$

This background corresponds to the direct product of the axially gauged coset $SU(2)/U(1)_a$ (or $SL(2,\mathbf{R})/U(1)_a$) and a free scalar field in the singular compactification-radius limit $r \to 0$ (which is equivalent to a non-compact free scalar field via the $r \to 1/r$ circle duality). The constant antisymmetric tensor in (4.8.12) can be safely dropped since the free scalar is non-compact.

At $R=\infty$ ($\alpha=\pi/2$) the background matrix is

$$E_{R=\infty}=\begin{pmatrix} 0 & 0 \\ 0 & \frac{1+\Sigma}{1-\Sigma} \end{pmatrix}. \tag{4.8.13}$$

This background corresponds to the direct product of the vectorially gauged coset $SU(2)/U(1)_v$ or $SL(2,\mathbf{R})/U(1)_v$ and a free scalar field at a compactification radius $r\to 0$ [39].

The $R=1$ ($\alpha=0$) point corresponds to the original WZNW model. Around this point, and for an infinitesimal deformation parameter $\delta\alpha$, the perturbed action is given by

$$S_{R=1+\delta R}=S_{R=1}+\frac{\delta R^2}{4\pi k}\int d^2z J\bar{J}\ , \tag{4.8.14}$$

where J and $\bar{J}$ are given in (4.8.6). This deformation can be extended along the full R-line. The $U(1)_L$ affine symmetry is generated by $\theta\to\theta-\epsilon$, $\tilde{\theta}\to\tilde{\theta}+\epsilon/R^2$ with the (R-dependent) chiral current

$$J(R)=k\,\frac{-(1-\Sigma)\partial\theta+(1+\Sigma)\partial\tilde{\theta}}{1+\Sigma+R^2(1-\Sigma)}\ , \tag{4.8.15}$$

and the $U(1)_R$ affine symmetry is generated by $\theta\to\theta+\epsilon$, $\tilde{\theta}\to\tilde{\theta}+\epsilon/R^2$ with the anti-chiral current

$$\bar{J}(R)=k\,\frac{(1-\Sigma)\bar{\partial}\theta+(1+\Sigma)\bar{\partial}\tilde{\theta}}{1+\Sigma+R^2(1-\Sigma)}\ . \tag{4.8.16}$$

Therefore, the worldsheet action at the point $R+\delta R$ is given by

$$S_{R+\delta R}=S_R+\frac{\delta R^2}{4\pi k}\int d^2z J(R)\bar{J}(R)\ , \tag{4.8.17}$$

and writing the integration measure for the perturbed σ-model as $\sqrt{G(1)}=\sqrt{G(R)}e^{\phi(R)}$ provides the variation of the dilaton (4.8.11).

We now arrive to the important point of this section. The Weyl transformation $J\to -J$ is given by a group rotation at the WZNW point, and therefore, it is a symmetry of the WZNW model. Consequently, the conformal perturbation of the WZNW model by $\delta\alpha J\bar{J}$ is equivalent, infinitesimally, to the conformal perturbation by $-\delta\alpha J\bar{J}$. Therefore, the points $\delta\alpha$ and $-\delta\alpha$ along the α-modulus correspond to the same CFT. In string theory we say that they are related by a spontaneously broken $\mathbf{Z}_2$ gauge transformation in the gauge group of the enhanced symmetry point.

The spontaneously broken discrete symmetry is a target space duality. This symmetry can be extended to finite α, giving rise to a $\mathbf{Z}_2$ element in the full $O(2,2,\mathbf{Z})$ duality group, represented

[39] The fact that at both boundaries $R=0,\infty$ the decoupled scalar is at zero radius is because in one limit the scalar is θ and in the other limit it is the dual field $\tilde{\theta}$.

by the matrix g_D

$$\begin{aligned} g_D &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & -\epsilon \\ 0 & I \end{pmatrix} \begin{pmatrix} e_2 & e_1 \\ e_1 & e_2 \end{pmatrix} \begin{pmatrix} I & \epsilon \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \end{pmatrix}, \end{aligned} \tag{4.8.18}$$

where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.8.19}$$

Here I is the 2-dimensional identity matrix, and ϵ is given in (4.8.8). Up to a similarity transformation, g_D is a factorized duality (2.4.29). The element g_D acts on E_R by (4.2.31) and gives

$$g_D(E_R) = E_{1/R}. \tag{4.8.20}$$

Therefore, duality takes the modulus R to its inverse $1/R$.

A particular consequence is that the $R = 0$ and $R = \infty$ boundary points correspond to the same CFT (to all orders in string perturbation theory). Therefore, the axial-vector duality of $SU(2)/U(1)$ and $SL(2,\mathbf{R})/U(1)$ is *exact*, and corresponds in string theory to a spontaneously broken gauge symmetry of some underlying string gauge algebra[40].

4.8.2 $J\bar{J}$ Deformations and Duality as a Gauge Symmetry for General Groups

Here we describe $J\bar{J}$ deformations of general WZNW models. These conformal perturbations are related to $O(d,d,\mathbf{R})$ transformations [168, 197, 139]. We start with a particular parametrization of a WZNW model on a group G. The group G can be semisimple. We explicitly indicate one of the levels k, while the others are hidden in the action. The relevant level is the one corresponding to the simple component of the group whose Cartan we are deforming. We parametrize $g \in G$ as follows: we chose an element in the Cartan sub-algebra, say T^1, and write it as

$$g = e^{i\theta_1 T^1} h(x) e^{i\theta_2 T^1}, \tag{4.8.21}$$

where $h \in G$ is independent of θ_1, θ_2. Now, by using the Polyakov-Wiegman formula [243] for the WZNW action one finds (see for example [196, 197])

$$S[x^a, \theta_1, \theta_2] = S_1 + S_a + S[x], \tag{4.8.22}$$

[40] The σ-models along the $R < 1$ half line can be obtained as the backgrounds of the axially gauged $SU(2) \times U(1)/U(1)_{diag}$ cosets, while the $R > 1$ backgrounds can be obtained by the vectorially gauged $SU(2) \times U(1)/U(1)_{diag}$ cosets [139]. The $R \to 1/R$ duality relates these axial cosets to their vector partners, and therefore, it is an axial-vector duality along the full R-modulus, and in particular at its boundaries.

$$S_1 = \frac{k}{2\pi}\int d^2z\left(\partial\theta_1\bar\partial\theta_1 + \partial\theta_2\bar\partial\theta_2 + 2\Sigma(x)\partial\theta_2\bar\partial\theta_1 + \Gamma^1_a(x)\partial x^a\bar\partial\theta_1 + \Gamma^2_a(x)\partial\theta_2\bar\partial x^a\right), \tag{4.8.23}$$

$$S_a = \frac{k}{2\pi}\int d^2z(\partial\theta_2\bar\partial\theta_1 - \partial\theta_1\bar\partial\theta_2), \tag{4.8.24}$$

$$S[x] = \frac{k}{2\pi}\int d^2z\ \Gamma_{ab}(x)\partial x^a\bar\partial x^b - \frac{1}{8\pi}\int d^2z\ \phi_0 R^{(2)}, \tag{4.8.25}$$

where $a = 1, ..., D-2$, $D = \dim G$, and ϕ_0 is a constant dilaton. The backgrounds $\Sigma(x), \Gamma^1(x), \Gamma^2(x)$ and $\Gamma(x)$ are independent of the coordinates θ_1, θ_2, and therefore, (4.8.22)-(4.8.25) is of the form (4.2.1) with $i, j = 1$ (up to an overall k factor). With the coordinates θ and $\tilde\theta$ defined in (4.8.2, 4.8.3), the action becomes

$$S[x^a, \theta, \tilde\theta] = \frac{1}{2\pi}\int d^2z(\partial\theta, \partial\tilde\theta, \partial x^a)\begin{pmatrix} E & F^2_b \\ F^1_a & F_{ab}\end{pmatrix}\begin{pmatrix}\bar\partial\theta \\ \bar\partial\tilde\theta \\ \bar\partial x^b\end{pmatrix} - \frac{1}{8\pi}\int d^2z\ \phi_0 R^{(2)}, \tag{4.8.26}$$

Here we have organized the background fields into a $D\times D$ background matrix with four blocks E, F^1, F^2 and F. The 2×2 background block E, the $(D-2)\times 2$ block F^2_b, the $2\times(D-2)$ block F^1_a, and the $(D-2)\times(D-2)$ block F_{ab} are given by

$$\begin{pmatrix} E & F^2_b \\ F^1_a & F_{ab}\end{pmatrix} = \frac{k}{2}\begin{pmatrix}\begin{pmatrix} 1-\Sigma & 1+\Sigma \\ -(1+\Sigma) & 1+\Sigma\end{pmatrix} & \begin{pmatrix}\Gamma^2_b \\ \Gamma^2_b\end{pmatrix} \\ \begin{pmatrix}-\Gamma^1_a & \Gamma^1_a\end{pmatrix} & 2\Gamma_{ab}\end{pmatrix}. \tag{4.8.27}$$

The action (4.8.26) is manifestly invariant under the $U(1)_L\times U(1)_R$ affine symmetry generated by the chiral current J and the anti-chiral current $\bar J$ given by

$$\begin{aligned} J &= \frac{k}{2}\left[-(1-\Sigma)\partial\theta + (1+\Sigma)\partial\tilde\theta + \Gamma^1_a\partial x^a\right], \\ \bar J &= \frac{k}{2}\left[(1-\Sigma)\bar\partial\theta + (1+\Sigma)\bar\partial\tilde\theta + \Gamma^2_a\bar\partial x^a\right]. \end{aligned} \tag{4.8.28}$$

In addition, there are extra $D-1$ chiral currents and $D-1$ anti-chiral currents, completing the affine $G_L\times G_R$ symmetry of the WZNW model[41].

We now deform the action (4.8.26) with the conformal perturbation $J\bar J$, where J and $\bar J$ are given in (4.8.28). The $J\bar J$ deformation is equivalent to the action of $g\in O(2,2,\mathbf{R})$ transformations on the background matrices E, F^1, F^2, F. The action of g in (4.2.26) on E is given

[41] Actually, one can bring the background in (4.8.22)-(4.8.25) into the form of eq. (4.2.1) [147] with $i, j = 1, ..., r = \mathrm{rank}\, G$ (up to an overall k factor). This can be done by explicitly parametrizing the Cartan torus dependence of the WZNW model as

$$g = e^{i\sum_{i=1}^r \theta^i_1 T^i} h(x^a) e^{i\sum_{i=1}^r \theta^i_2 T^i},$$

where $\{T^i | i = 1, ..., r\}$ is a basis in the Cartan sub-algebra, $\mathrm{Tr}(T^iT^j) = \eta^{ij}$, and $h\in G$ is independent of θ^i_1, θ^i_2. In this form the r chiral currents and r anti-chiral currents corresponding to the left-handed and right-handed Cartan tori are manifest.

by (4.2.31). Here we need also the action of g on the background blocks F^1, F^2, F given in eq. (4.2.30) [147]:

$$g(F^1) = F^1(cE+d)^{-1}, \qquad g(F^2) = (a - E'c)F^2,$$
$$g(F) = F - F^1(cE+d)^{-1}cF^2, \tag{4.8.29}$$

where a, c, d are defined in (4.2.26), and E' is given in (4.2.31).

The modulus line of $J\bar{J}$ conformal perturbations is generated by acting on the background (4.8.27) with the one-parameter family of $O(2,2,\mathbf{R})$, g_α, given in (4.8.7). Using eqs. (4.2.31, 4.8.29) one finds

$$\frac{1}{k} g_\alpha(E) \equiv E_{R(\alpha)}, \tag{4.8.30}$$

$$\frac{1}{k} g_\alpha(F^1) \equiv F^1_{R(\alpha)} = \frac{(1+\Sigma)^{-1}\Gamma^1}{1+R^2\frac{1-\Sigma}{1+\Sigma}} \left(-1, R^2\right), \tag{4.8.31}$$

$$\frac{1}{k} g_\alpha(F^2) \equiv F^2_{R(\alpha)} = \frac{(1+\Sigma)^{-1}\Gamma^2}{1+R^2\frac{1-\Sigma}{1+\Sigma}} \begin{pmatrix} 1 \\ R^2 \end{pmatrix}, \tag{4.8.32}$$

$$\frac{1}{k} g_\alpha(F) \equiv F_{R(\alpha)} = \Gamma + \frac{(R^2-1)(1+\Sigma)^{-1}\Gamma^1\Gamma^2}{2(1+R^2\frac{1-\Sigma}{1+\Sigma})}, \tag{4.8.33}$$

where E_R and $R(\alpha)$ are given in (4.8.9, 4.8.10).

The constant dilaton ϕ_0 in the WZNW background (4.8.25) transforms under g_α by eq. (4.8.11). For more details see sections 4.2, 4.3.

From eqs. (4.8.9, 4.8.31, 4.8.32, 4.8.33) we see that, up to an overall k factor, the backgrounds are parametrized by a positive real number $R^2 = (1 + k\tan\alpha)^2$. As for the $SU(2)$ case, the parameter R can be interpreted, geometrically, as the radius of the corresponding circle in the deformed Cartan torus.

As for the $SU(2)$ or $SL(2,\mathbf{R})$ cases, the whole Cartan subalgebra survives along the deformation. One can find that the explicit form of the (R-dependent) chiral and anti-chiral currents we perturb with is

$$J(R) = k\, \frac{-(1-\Sigma)\partial\theta + (1+\Sigma)\partial\tilde{\theta} + \Gamma^1_a \partial x^a}{1+\Sigma+R^2(1-\Sigma)}\,, \tag{4.8.34}$$

$$\bar{J}(R) = k\, \frac{(1-\Sigma)\bar{\partial}\theta + (1+\Sigma)\bar{\partial}\tilde{\theta} + \Gamma^2_a \bar{\partial} x^a}{1+\Sigma+R^2(1-\Sigma)}\,, \tag{4.8.35}$$

and it can be verified that eq. (4.8.17) is still valid[42].

[42] The deformation described here is true for any background with a chiral current and an anti-chiral current; it is only for the purpose of discussing gauge symmetries that we assume that a special point on the modulus line (in our notation $R = 1$) is a WZNW model.

We are now ready to discuss target space duality in the R-line of conformal deformations. Remarkably, the action of the $O(2,2,\mathbf{Z})$ element g_D (given in eq. (4.8.18)) on the backgrounds E, F^1, F^2, F in eq. (4.8.27), gives a simple transformation of the modulus parameter R. By straightforward calculations, using the transformations given in (4.2.31, 4.8.29), one finds that

$$g_D \left(\begin{pmatrix} E_R & F^2_R \\ F^1_R & F_R \end{pmatrix} \right) = \begin{pmatrix} E_{1/R} & F^2_{1/R} \\ F^1_{1/R} & F_{1/R} \end{pmatrix}. \tag{4.8.36}$$

Therefore, duality relates the σ-model background at the modulus point R with the background at the modulus parameter $1/R$.

It is now easy to generalize the consequences described for the $SU(2)$ and $SL(2,\mathbf{R})$ cases in subsection 4.8.1. At the fixed point $R=1$ the CFT has an enhanced affine symmetry $G_L \times G_R$: it is the original WZNW model. Infinitesimally around the enhanced symmetry point, duality corresponds to the transformation $\alpha \to -\alpha$. This transformation can be achieved by a Weyl rotation in G_L (or G_R) that reflects $J \to -J$ (or $\bar{J} \to -\bar{J}$). Duality is related to a Weyl reflection and is, therefore, a spontaneously broken gauge symmetry in the gauge algebra of the associated string theory.

The duality transformation interchanges R with $1/R$ and, in particular, identifies the two boundaries at $R=0$ and $R=\infty$. Let us show that these boundaries correspond, respectively, to the direct product of the cosets $G/U(1)_a$ and $G/U(1)_v$ with a free, non-compact scalar field.

At $R=0$ ($\alpha = \tan^{-1}(-\frac{1}{k})$) the background matrix is

$$\begin{aligned} E_{R=0} = \begin{pmatrix} \frac{1-\Sigma}{1+\Sigma} & 1 \\ -1 & 0 \end{pmatrix} &= \begin{pmatrix} \frac{1-\Sigma}{1+\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{mod } \Theta - \text{shift}, \\ F^1_{R=0} = (1+\Sigma)^{-1}\Gamma^1(-1,0), &\qquad F^2_{R=0} = (1+\Sigma)^{-1}\Gamma^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ F_{R=0} &= \Gamma - \frac{1}{2}(1+\Sigma)^{-1}\Gamma^1\Gamma^2. \end{aligned} \tag{4.8.37}$$

This background corresponds, as shown in section 4.2 [246, 147], to the direct product of the axially gauged coset $G/U(1)_a$ and a free scalar field in the compactification-radius limit $r \to 0$.

At $R=\infty$ ($\alpha = \pi/2$) the background matrix is

$$\begin{aligned} E_{R=\infty} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1+\Sigma}{1-\Sigma} \end{pmatrix}, &\qquad F_{R=\infty} = \Gamma + \frac{1}{2}(1-\Sigma)^{-1}\Gamma^1\Gamma^2, \\ F^1_{R=\infty} = (1-\Sigma)^{-1}\Gamma^1(0,1), &\qquad F^2_{R=\infty} = (1-\Sigma)^{-1}\Gamma^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{4.8.38}$$

This background corresponds to the direct product of the vectorially gauged coset $G/U(1)_v$ and a free scalar field at $r \to 0$ (see section 4.2).

Therefore, we have obtained that axial-vector duality is exact in general, since the end-points are equivalent theories, analogously to the $SU(2)$ and $SL(2,\mathbf{R})$ cases.

4.9 Topology Change and Duality in String Theory

In this section we discuss topology change in string theory and its relation with target space duality. This issue will be addressed in particularly simple examples. Here we present models in which connected paths in moduli space exist along which the topology of the background metric changes. It would be interesting to construct a CFT corresponding to a time dependent background, where topology change occurs as a dynamical process.

The simplest example of topology change in string theory is observed in the moduli space of $c = 1$ backgrounds in figure 3.A. At the point where the circle compactification line meets the orbicircle line – the orbicircle self-dual point $R_o = 1$ – the topology may change from a circle to a segment. Moreover, at the $SU(2)$ enhanced symmetry point – the circle self-dual point $R_c = 1$ – the σ-model description of the CFT can be given either in terms of a circle or, alternatively, in terms of the three-dimensional sphere background of the $SU(2)_1$ WZNW model. In these examples one may connect semiclassical (i.e. large) backgrounds of different topology. However, the actual topology change occurs due to "stringy schizophrenia" of backgrounds at the Planck scale: in some instances the string cannot perceive in which world it lives.

Let us now discuss examples where topology change occurs through a different mechanism. Namely, a single background is associated to each point of the connected path in the moduli space: no doubling of the kind described above is invoked, rather, the topology changes when classical curvature singularities arise. We will describe the geometry along the R-line of section 4.8 and a topology change in the extended moduli space of $SU(2)$ or $SL(2,\mathbf{R})$ [139]. Moreover, we will show that there is a topology change in the moduli space of the cosmological string solutions discussed in section 4.8.

We present at first the description of the target space geometry along the R-line of marginal deformations of the $SU(2)$ WZNW model. The σ-model metric of the deformed $SU(2)$ (4.8.9) (in the coordinates θ, $\tilde{\theta}$, x) is given by

$$G(R) = k \begin{pmatrix} \frac{\tan^2 x}{1+R^2\tan^2 x} & 0 & 0 \\ 0 & \frac{R^2}{1+R^2\tan^2 x} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.9.1}$$

The scalar curvature $\mathcal{R}$ is

$$\mathcal{R} = -\frac{2}{k}\frac{2 - 5R^2 + 2(R^4-1)\sin^2 x}{(1+(R^2-1)\sin^2 x)^2} . \tag{4.9.2}$$

The manifold is regular except at the end-points, where

$$\mathcal{R}(R=0) = -\frac{4}{k\cos^2 x} \ , \ \mathcal{R}(R=\infty) = -\frac{4}{k\sin^2 x} . \tag{4.9.3}$$

At $R = 1$ we get the constant curvature of S^3, $\mathcal{R} = 6/k$. The metric $G(R)$ has no conical singularities along the R-line.

Let us note that the geometrical data (metric, curvature, etc.) are invariant under the $R \to 1/R$ duality, together with the coordinate transformation $x \to \pi/2 - x$. In particular, the volume of the manifold as a function of R,

$$V(R) \sim \frac{R \log R}{R^2 - 1}, \tag{4.9.4}$$

is duality invariant: $V(R) = V(1/R)$. The volume vanishes only at the boundaries of moduli space, namely, at $R = 0, \infty$.

Along the $0 < R < \infty$ deformations line of the $SU(2)$ WZNW σ-model, the topology of the background space is of the three sphere. However, at the boundaries ($R = 0, \infty$), the topology is changed to that of a product of a two-disc (corresponding to the $SU(2)/U(1)$ conformal background) with a circle whose radius shrinks to 0.

For $SL(2, \mathbf{R})$, the trigonometric functions in (4.9.2) are replaced by the corresponding hyperbolic functions. Here the manifold has a curvature singularity for $0 \leq R < 1$; similar remarks apply to its Euclidean version, the 3-d hyperboloid. The background at the boundary is the direct product of a degenerated circle and a semi-infinite "cigar" (at $R \to \infty$) with the topology of the disk, or the infinite "trumpet" (at $R \to 0$) with the topology of a cylinder; these correspond to the dual pair of the 2-d Euclidean black-hole backgrounds (see section 4.5 and figure 1.F).

A topology change at the boundary of moduli space is not surprising. However, the R-line of deformations is not the full story in the moduli space of G WZNW σ-models. Indeed, any conformal σ-model with d Abelian symmetries can be transformed to new conformal backgrounds by $O(d, d, \mathbf{R})$ rotations (see section 4.3)[43]. In this larger moduli space of σ-models, there are other interesting deformation lines, along which the topology might change.

For example, in the moduli space of the $SU(2)$ WZNW σ-model, there is a one-parameter family of $O(2, 2, \mathbf{R})$ rotations

$$\tilde{g}_\alpha = \begin{pmatrix} I & k\epsilon \\ 0 & I \end{pmatrix} \begin{pmatrix} C(\alpha) & S(\alpha) \\ S(\alpha) & C(\alpha) \end{pmatrix} \begin{pmatrix} I & -k\epsilon \\ 0 & I \end{pmatrix} \tag{4.9.5}$$

($\epsilon, C(\alpha), S(\alpha)$ are given in (4.8.8)), that generate the backgrounds

$$\tilde{g}_\alpha(E) = \frac{k}{\Delta} \begin{pmatrix} 1 - \Sigma & B \\ -B & 1 + \Sigma \end{pmatrix}, \tag{4.9.6}$$

where

$$\begin{aligned} \Sigma = \cos 2x \ , \qquad & \Delta = \cos^2 \alpha (1 + \Sigma) + (\cos \alpha + k \sin \alpha)^2 (1 - \Sigma) \ , \\ B \ = \ & \frac{1}{k} \sin(2\alpha) - \left[\cos(2\alpha) + \frac{k}{2} \sin(2\alpha) \right] (1 - \Sigma) + \Delta \ . \end{aligned} \tag{4.9.7}$$

[43] Recall that in the bosonic string, the $O(d, d, \mathbf{R})$ rotations give only the leading order in α' of the conformal backgrounds, but there exist higher order corrections that make them exact [147, 142, 197, 139].

The dilaton transforms by eq. (4.8.11).

The (α-dependent) metric is given by the line element

$$\begin{aligned} ds^2(\alpha) &= \frac{2k}{\Delta(\alpha)}[\sin^2 x d\theta^2 + \cos^2 x d\tilde{\theta}^2] + k dx^2, \\ \Delta(\alpha) &= 2\cos^2\alpha\cos^2 x + 2(\cos\alpha + k\sin\alpha)^2\sin^2 x. \end{aligned} \tag{4.9.8}$$

At the point $\alpha = 0$ the background includes a metric of the $SU(2)_k$ group manifold S^3 (as well as an antisymmetric background). Along the line $0 < \alpha < \pi/2$ the background includes the metric (4.9.8) with the topology of S^3 (as well as an antisymmetric background and a dilaton field). At the point $\alpha = \pi/2$ the background metric is

$$ds^2(\alpha = \pi/2) = \frac{1}{k} d\theta^2 + \frac{1}{k}\cot^2 x d\tilde{\theta}^2 + k dx^2. \tag{4.9.9}$$

At this point the manifold has a topology of $D_2 \times S^1_{1/k}$, where D_2 is a two-disc and $S^1_{1/k}$ is a circle with radius $r^2 = 1/k$. One may continue to deform this theory by, for example, changing the compactification radius r of the free scalar field θ.

It is remarkable that (for integer k) the neighborhood of the point $\alpha = \pi/2$ is mapped to the neighborhood of the point $\alpha = 0$ by an element of $O(2,2,\mathbf{Z})$ [246, 147, 205], namely, by a target space generalized duality. Therefore, *a region in the moduli space, where a topology change occurs, is mapped to a region where there is no topology change at all.* A similar phenomenon happens for more complicated examples in the moduli space of Calabi-Yau compactifications [13, 287].

At this point, let us make two comments:

(a) The σ-models along the α–line (4.9.8) have conical singularities. Therefore, to make sense of the CFTs along the α–line one should understand CFTs corresponding to backgrounds with (non-orbifold) conical singularities.

(b) After the topology is changed at $\alpha = \pi/2$, one cannot get rid of the curvature singularity encountered at that point simply by deforming the compactification radius of the free scalar field θ.

To cure both problems, one should look at the moduli space of WZNW σ-models in higher dimensions.

Indeed, an even more interesting example of topology change occurs in the moduli space of the $SU(2)\times SL(2,\mathbf{R})$ WZNW model. Here one can follow deformation lines along which there is no scalar curvature singularity, except at the point where the topology is changed. Explicitly, one first deforms $SU(2)\times SL(2,\mathbf{R})$ to the background of the coset $\left[\frac{SU(2)}{U(1)}\times\frac{SL(2,\mathbf{R})}{U(1)}\times U(1)\right]\times U(1)'$, where the background of the coset in the brackets is given in (4.7.14) together with (4.7.8), and

with $a = b = c = 0$. Now, one can rotate the background by $O(3,3,\mathbf{R})$ transformations into the backgrounds (4.7.14) with a, b, c simultaneously non-zero. The resulting backgrounds correspond to the product of cosmological solutions in the presence of a gauge field (4.7.14) times the circle $U(1)'$. As shown in section 4.7 [142], these backgrounds have no scalar curvature singularities.

Finally, and as before, the point $a = b = c = 0$ is related by $O(3,3,\mathbf{Z})$ target space dualities to points $a, b, c \in \mathbf{Z}$ (see the end of section 4.7) where no topology change occurs.

5 Worldsheet/Target-Space Duality

A classical action describing the propagation of an extended object of dimension d in a space-time of dimension D is described classically by a d-dimensional world-manifold field theory, with D target space coordinates. For strings, the $d = 2$ worldsheet theory can actually be quantized. In this section we present some mathematical identities. They are intriguing as they may suggest that the target space and the worldsheet are actually part of a larger structure.

In the following it is shown [140] that there exist relations between pairs (Σ, T), where Σ denotes the worldsheet and T denotes the target space. It is shown that in some cases the "classical"[44] part of the partition function describing pairs (Σ, T) is related to the classical part of the pair (T, Σ), where the worldsheet and target space are interchanged. Most of the discussion in this section deals with the bosonic string. The classical part of the pairs (Σ, T) and (T, Σ) is manifestly identical if the following conditions are satisfied. Either the target space is "almost complex" or the period matrix τ describing the Riemann surface is "real." The terms "almost complex" and "real" in this context will be defined below. This symmetry relates the dimension of the target space of a string theory to the order of interaction, namely, the genus of the worldsheet Riemann surface. Duality relations between spaces of different dimensions were discussed and utilized in the study of statistical mechanics systems [9].

Here we present the mathematical identities in an attempt to learn about the symmetries of string theories. We first consider toroidal bosonic compactifications. As an immediate consequence of the identities, we will prove that $Sp(2d, \mathbf{Z}) \subset O(2d, 2d, \mathbf{Z})$ is a symmetry of the genus-g partition functions for almost complex toroidal backgrounds in $2d$ dimensions. Moreover, we will prove that $O(d, d, \mathbf{Z}) \subset Sp(4d, \mathbf{Z})$ is a symmetry of the genus-g partition functions with real period matrices and any d-dimensional flat target space. A proof that $O(d, d, \mathbf{Z})$ is a symmetry to all orders and for any Riemann surface is needed in order to show that $O(d, d, \mathbf{Z})$ is a symmetry valid to all orders in string perturbation theory. This proof can be obtained by combining the methods described in section 2 with the results of this section.

5.1 Worldsheet/Target-Space Duality for Toroidal Bosonic Compactifications

Let $E = G + B$ be a $2d \times 2d$ background matrix (see section 2) [140] of the form

$$E = \begin{pmatrix} \sigma_2 & \sigma_1 \\ -\sigma_1 & \sigma_2 \end{pmatrix}, \tag{5.1.1}$$

[44] In this section "classical" refers to the partition function corresponding to the zero-mode Hamiltonian in eq. (2.4.11).

where σ_1 and σ_2 are real, symmetric $d \times d$ matrices, and σ_2 is positive definite. The metric G and antisymmetric background B are, therefore,

$$G = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}. \tag{5.1.2}$$

Denote by I_n the n-dimensional identity matrix, and by J_n the standard $2n$-dimensional antisymmetric matrix,

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad J_n^2 = -I_{2n}, \tag{5.1.3}$$

and define the complex $d \times d$ matrix σ :

$$\sigma = \sigma_1 + i\sigma_2. \tag{5.1.4}$$

The specific form (5.1.1) of E means that the target space is "almost complex" in the sense that $E \equiv -i\sigma$, i.e. :

$$G = (\Re E) \otimes I_2, \qquad B = (-\Im E) \otimes J. \tag{5.1.5}$$

In the loop expansion of string theory one sums over all mappings from a genus-g Riemann surface $\sum_g$ to the target space T^{2d}. In a canonical basis of cycles, a_i, b_i, where $i = 1, \ldots, g$ (see section 2.3 and figure 2.A), the Riemann surface is described by the period matrix $\tau = \tau_1 + i\tau_2$. The $g \times g$ matrices τ_1 and τ_2 are real and symmetric, and τ_2 is positive definite. The classical piece of the multi-loop path integral will be denoted by $Z_{class}(\sigma, \tau)$. In what follows it is observed [140] that

$$Z_{class}(\sigma, \tau) = Z_{class}(\tau, \sigma), \tag{5.1.6}$$

namely, the classical piece of the mapping from a Riemann surface $\Sigma_g(\tau)$ to a target space $T^{2d}(\sigma)$ is equivalent to the mapping from a Riemann surface $\Sigma_d(\sigma)$ to a target space $T^{2g}(\tau)$.[45]

The result (5.1.6) can be useful in proving the invariance of genus-g partition functions under generalized target space duality. Notice that there is an isomorphism between $Sp(2g, \mathbf{Z})$ (the group of modular transformations of the Riemann surface) generated by [232]

$$\begin{aligned} T &: \tau \to \tau + b, \qquad b \ \text{ symmetric, integral,} \\ S &: \tau \to -1/\tau, \\ A &: \tau \to m^t \tau m, \qquad m \in GL(g, \mathbf{Z}), \end{aligned} \tag{5.1.7}$$

and the symmetry transformations on the target space moduli generated by

$$\begin{aligned} T' &: \sigma \to \sigma + b, \qquad b \ \text{ symmetric, integral,} \\ S' &: \sigma \to -1/\sigma, \\ A' &: \sigma \to m^t \sigma m, \qquad m \in GL(d, \mathbf{Z}). \end{aligned} \tag{5.1.8}$$

[45] This result was observed first in the $d = 1$, $g = 1$ case [81] (2.4.62). In this case it can be generalized to a triality while reintroducing the target space complex structure.

As discussed in section 2, invariance under T' reflects the fact that the B terms serve as two dimensional theta parameters. Invariance under S' is the inversion duality symmetry of the closed string with $2d$ compactified dimensions. The action A' on the background σ is equivalent to a change of the basis of the lattice defining the target space torus. There is, therefore, *an intriguing connection between target space duality and worldsheet modular invariance.* The duality symmetry $E \to 1/E$ (2.4.38) ($\sigma \to -1/\sigma$) is a simple consequence of the symmetry $\tau \leftrightarrow \sigma$. In fact, this is a way to prove that inversion duality is an exact symmetry to all orders in loop expansion. Moreover, it also follows that the subgroup $Sp(2d, \mathbf{Z}) \subset O(2d, 2d, \mathbf{Z})$ is a symmetry to all orders in loop expansion, for almost complex toroidal backgrounds. A proof that $O(d, d, \mathbf{Z})$ is a symmetry to all orders, and for any toroidal backgrounds, is straightforward.

To prove eq. (5.1.6) we work in the notation of section 2. The closed bosonic string action for the compactified dimensions (setting the dilaton to zero, for the time being) is

$$S = \frac{1}{4\pi} \int d^2\sigma (\sqrt{h} h^{\alpha\beta} G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu), \tag{5.1.9}$$

where $\alpha, \beta = 0, 1$; $\mu, \nu = 1, \ldots, 2d$; σ_0, σ_1 are the world-sheet parameters; $h^{\alpha\beta}$ is the worldsheet metric; $h = \det h_{\alpha\beta}$; $\epsilon^{\alpha\beta}$ is the standard 2-dimensional antisymmetric tensor. Here, the compact space is a $2d$-dimensional torus $T^{2d} = \mathbf{R}^{2d}/\pi\Lambda^{2d}$; the lattice Λ^{2d} is spanned by the basis $\{e_i^\mu \mid i = 1, \ldots, 2d\}$. Thus X^μ is identified with $X^\mu + \pi n^i e_i^\mu$, where n^i are integers. Let us define the constant metric G_{ij} and the constant antisymmetric tensor B_{ij} by

$$\begin{aligned} G_{\mu\nu} &= 2G_{ij}(e^{i*})_\mu (e^{j*})_\nu, \\ B_{\mu\nu} &= 2B_{ij}(e^{i*})_\mu (e^{j*})_\nu, \end{aligned} \tag{5.1.10}$$

where $\{e^{i*} \mid i = 1, \ldots, 2d\}$ is the basis dual to $\{e_i\}$. In the orthonormal gauge, $h_{\alpha\beta} = e^{\phi(z,\bar{z})}\delta_{\alpha\beta}$, the action may be written as

$$S = \frac{1}{2\pi} \int d^2z E_{ij} \partial X^i \bar{\partial} X^j, \tag{5.1.11}$$

where $E = G + B$; $X^i \equiv X^\mu (e^{i*})_\mu$. This action depends on the period matrix τ through the complex structure, used to define z and $\bar{z}$.

The classical piece of the genus-g partition function gives a contribution [8, 6, 136, 140]

$$\begin{aligned} Z_{class}(G, B, \tau) &= (\det G)^{\frac{g}{2}} \sum_{n,m} \exp\{-\pi [n^{ia} (\tau_2^{-1})_{ab} G_{ij} n^{jb} \\ &\quad + m^{ia} (\tau_2 + \tau_1 \tau_2^{-1} \tau_1)_{ab} G_{ij} m^{jb} \\ &\quad - 2m^{ia} (\tau_1 \tau_2^{-1})_{ab} G_{ij} n^{jb} + 2i n^{ia} B_{ij} m^{ja}]\}, \end{aligned} \tag{5.1.12}$$

where $a, b = 1, \ldots, g$; n and m are $2dg$-dimensional vectors with integer components. (This is a generalization of the result presented for circle compactifications in section 2.3). The normalization is such that $Z_{class} = \mathrm{Tr}(\exp(-\tau_2 H + i\tau_1 P))$ in the one-loop case ($g = 1$) (H is

the Hamiltonian and P is the worldsheet momentum). In the path integral formulation at one loop, one gets $(\det G)^{1/2}$ by integrating the zero modes X_0: it is the volume of the target space.

For G and B of the form (5.1.2) one gets

$$\begin{aligned} Z_{class}(\sigma,\tau) &= (\det\sigma_2)^g \sum_{m_1,m_2,n_1,n_2} \exp\{-\pi[n_1\tau_1^{-1}\sigma_2 n_1 + n_2\tau_2^{-1}\sigma_2 n_2 \\ &+m_1(\tau_2+\tau_1\tau_2^{-1}\tau_2)\sigma_2 m_1 + m_2(\tau_2+\tau_1\tau_2^{-1}\tau_1)\sigma_2 m_2 \\ &-2m_1\tau_1\tau_2^{-1}\sigma_2 n_1 - 2m_2\tau_1\tau_2^{-1}\sigma_2 n_2 \\ &+2in_1\sigma_1 m_2 - 2in_2\sigma_1 m_1]\}, \end{aligned} \tag{5.1.13}$$

where n_1, n_2, m_1, m_2 are dg-dimensional vectors of integer components. A Poisson resummation on m leads to

$$\begin{aligned} Z_{class}(\sigma,\tau) &= (\det\tau_2)^d \sum_{m_1,m_2,k_1,k_2} \exp\{-\pi[k_1\sigma_2^{-1}\tau_2 k_1 + k_2\sigma_2^{-1}\tau_2 k_2 \\ &+m_1(\sigma_2+\sigma_1\sigma_2^{-1}\sigma_1)\tau_2 m_1 + m_2(\sigma_2+\sigma_1\sigma_2^{-1}\sigma_1)\tau_2 m_2 \\ &+2m_1\sigma_1\sigma_2^{-1}\tau_2 k_2 - 2m_2\sigma_1\sigma_2^{-1}\tau_2 k_1 \\ &-2ik_1\tau_1 m_1 - 2ik_2\tau_1 m_2]\}. \end{aligned} \tag{5.1.14}$$

After the resummation the partition function can be written in a more compact form:

$$Z_{class}(\sigma,\tau) = (\det\tau_2)^d \sum_{m,k} \exp\{i\pi[P_R^a\tau_{ab}P_R^b - P_L^a\bar{\tau}_{ab}P_L^b]\}, \tag{5.1.15}$$

where, as in (2.4.12),

$$P_R^a = (k^a - m^a E^t)e^*, \qquad P_L^a = (k^a + m^a E)e^*. \tag{5.1.16}$$

Identifying (k_1,k_2,m_1,m_2) in the sum (5.1.14) with $(n_1,n_2,-m_2,m_1)$ in the sum (5.1.13) we get the proof of (5.1.6):

$$Z_{class}(\sigma,\tau) = Z_{class}(\tau,\sigma).$$

We recall that eq. (5.1.6) is proved for any element τ of the Siegel upper-half-space (the space of symmetric, complex $g\times g$ matrices with positive definite imaginary part). Not all of them correspond to period matrices of Riemann surfaces. The fact that eq. (5.1.6) is correct for any τ suggests that there might be a physical meaning to the partition function of a flat target space for any τ.

The target space structure in eq. (5.1.6) is restricted to a background matrix of the form σ (eq. (5.1.2) and eq. (5.1.5)). However, one can work with a general $d\times d$ background $E=G+B$ and a subset of the $2g$-dimensional Siegel upper-half-space of the form

$$\tau_2 = \begin{pmatrix} T_2 & 0 \\ 0 & T_2 \end{pmatrix}, \qquad \tau_1 = \begin{pmatrix} 0 & T_1 \\ -T_1 & 0 \end{pmatrix}, \tag{5.1.17}$$

where T_2 is a $g \times g$ positive definite symmetric matrix and T_1 is a $g \times g$ antisymmetric matrix. This subset generates a space isomorphic to $O(g,g,\mathbf{R})/(O(g,\mathbf{R}) \times O(g,\mathbf{R}))$. In this case, $\tau_1 + i\tau_2$ is equal to $J \otimes (T_2 + T_1)$, where J is the complex structure ($J^2 = 1$). The matrix $(T_2 + T_1)$ is real and in this sense τ is "real." Defining $T = T_2 + T_1$, one finds

$$Z_{class}(E,T) = Z_{class}(T,E), \tag{5.1.18}$$

namely, the classical partition function of a flat background described by E, and a period matrix τ described by T, is the same as that of a background described by T on a Riemann surface described by E. The subgroup of the genus-$2g$ modular group that keeps τ real is $O(g,g,\mathbf{Z}) \subset Sp(4g,\mathbf{Z})$. Therefore, as an immediate consequence of (5.1.18), we get that the target space duality group, $O(d,d,\mathbf{Z})$, is a symmetry to all orders, when the period matrices are real.

If the target space is "almost-complex" and the period matrix is "real" then there is a triality relation

$$Z_{class}(E_g, \tau_{4d}) = Z_{class}(E_{2d}, \tau_{2g}) = Z_{class}(E_{4g}, \tau_d), \tag{5.1.19}$$

where τ_a and E_b stand for the relevant $a \times a$ period matrix and the $b \times b$ background matrix, respectively. It is generalized to a multiality if the period matrix and the background matrix have a higher "complex-real" structure. For example, let

$$\begin{aligned} E_{2^n d} &= \gamma \otimes I_{2^n}, \\ \tau_{2^m g} &= it \otimes I_{2^m}, \end{aligned} \tag{5.1.20}$$

where γ (t) are $d \times d$ ($g \times g$) positive-definite symmetric matrices. Then there is a $(m+n+1)$-ality symmetry

$$\begin{aligned} \ldots &= Z(t \otimes I_{2^{m+1}}, \gamma \otimes I_{2^{n-1}}) = Z(\gamma \otimes I_{2^n}, t \otimes I_{2^m}) \\ &= Z(t \otimes I_{2^{m-1}}, \gamma \otimes I_{2^{n+1}}) = Z(\gamma \otimes I_{2^{n+2}}, t \otimes I_{2^{m-2}}) = \ldots \end{aligned} \tag{5.1.21}$$

The reduction in the number of handles on the worldsheet gives birth to copies of the target space γ, and vice versa. In case $m = 1$, $n = 0$ or $m = 0$, $n = 1$ we get the two possibilities of duality symmetry mentioned before (eqs. (5.1.6) and (5.1.18)). If $m = n = 1$ then we get the triality symmetry of eq. (5.1.19).

Additional observations on the target space - worldsheet coupling can be obtained by the following considerations. The exponent in equation (5.1.12) can be written as

$$(n - \bar{\tau}m)^t \tau_2^{-1} G (n - \tau m) + 2in^t B m = N^t (H_1 G + i H_2 B) N, \tag{5.1.22}$$

where we group the n and m into a column vector

$$N = \begin{pmatrix} n \\ m \end{pmatrix}, \tag{5.1.23}$$

and the matrices H_1 and H_2 are defined to be

$$H_1 = \begin{pmatrix} \tau_2^{-1} & -\tau_2^{-1}\tau_1 \\ -\tau_1\tau_2^{-1} & \tau_2 + \tau_1\tau_2^{-1}\tau_1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{5.1.24}$$

The complex matrix $H = H_1 + iH_2$ is the natural Hermitian form of the Jacobian torus [232]. A Riemann surface with g handles is described by a g-dimensional complex torus (called the "Jacobian torus")

$$J^g = \mathbf{C}^g/\Lambda, \qquad \Lambda = \mathbf{Z}^g \otimes \tau\mathbf{Z}^g. \tag{5.1.25}$$

Here Λ is the lattice with vectors $\pi_R = n - \tau m$, where n, m are g-vectors with integer components. The Hermitian form $\bar{\pi}_R\tau_2^{-1}\pi_R$ ($\bar{\pi}_R = n - \bar{\tau}m$) is related to H as

$$\bar{\pi}_R\tau_2^{-1}\pi_R = N^t H N. \tag{5.1.26}$$

The vectors π_R ($\pi_L = n + \tau m$) are "equivalent" to the "momenta" P_R (P_L) of eq. (5.1.16) (τ is replaced by E and J stands for i). Changing τ to σ in eq. (5.1.24), H_1 is nothing but the metric on the phase space $Z = (X', P)$ of the compactified string theory [146] (2.4.17). Therefore, a close relation between a genus-g Riemann surface and a $2g$-dimensional toroidal compactification is suggested.

It is interesting to notice that the metric of the target space, G, "feels" the symmetric piece of the Jacobian torus Hermitian form, H_1, while the antisymmetric background, B, "feels" only the antisymmetric piece, H_2. The constant dilaton background, ϕ, is coupled on the worldsheet to the two-dimensional scalar curvature $R^{(2)}$. The 2-d integration of $R^{(2)}$ leads to $g - 1$, where g is the genus. There exists a functional of ϕ, leading to $d - 2$ (or $d - 26$ with the ghost sector); that is the beta function of the dilaton. The full set of constant parameters we are dealing with is $(G, B, \phi, \tau_1, \tau_2)$. The parameter ϕ is "neutral" under the exchange of E with τ. However, ϕ might transform to some ϕ' in order to keep the overall factor correct (this is also true for the $O(d, d, \mathbf{Z})$ duality group symmetry on target space in higher genus). Indeed, it is the transformation of the dilaton under duality in eq. (4.2.37) which provides the correct factors. (For a detailed study of the transformation of the dilaton, we refer the reader to subsection 4.2.4).

Having pointed out similarities between the worldsheet and target space moduli, one can go a step further and attempt to view both as part of a larger space [221]. This can be approached by considering [221, 242] the symmetries of the classical partition function of a string moving in a general background E on a worldsheet manifold described by τ. The classical partition function can be written as [221]:

$$Z_{class}(E, \tau) = \sum_{n,m \in \mathbf{Z}^{dg}} \exp\left\{-\pi\left[(n^t, m^t)\begin{pmatrix} G \otimes (\tau_2 + \tau_1\tau_2^{-1}\tau_1) & G \otimes (\tau_1\tau_2^{-1}) + iB \otimes I \\ G \otimes (\tau_2^{-1}\tau_1) - iB \otimes I & G \otimes \tau_2^{-1} \end{pmatrix}\begin{pmatrix} n \\ m \end{pmatrix}\right]\right\}. \tag{5.1.27}$$

After performing a Poisson resummation it assumes the form

$$Z_{class}(E,\tau) = \sum_{n,k\in\mathbf{Z}^{dg}} \exp\left\{-\pi\left[(n^t,k^t)\begin{pmatrix} \tau_2\otimes(G-BG^{-1}B) & -\tau_2\otimes(BG^{-1}) - i\tau_1\otimes I \\ \tau_2\otimes(G^{-1}B) - i\tau_1\otimes I & \tau_2\otimes G^{-1}\end{pmatrix}\begin{pmatrix} n \\ k\end{pmatrix}\right]\right\}. \tag{5.1.28}$$

Both matrices in the exponents of (5.1.27) and (5.1.28) are symmetric, and their imaginary part is positive definite. As such, they are elements of the Siegel upper-half space of dimension $2dg$. One can define a map

$$E_d \otimes H_g \to H_{2dg}, \tag{5.1.29}$$

where E_d is a d-dimensional background and H_g is the one defined in eq. (5.1.24) for genus g. The map takes E and τ into the matrix appearing in the exponent of (5.1.27). One already knows that the group $O(d,d,\mathbf{Z})$ acts on E and the group $Sp(2g,\mathbf{Z})$ acts on τ, and both groups are symmetries of the partition function. Under the embedding (5.1.29), both groups are mapped into some subgroups of $Sp(4dg,\mathbf{Z})$, which is the modular group of H_{2dg}. This is the symmetry of the partition function. A generic element in $Sp(4dg,\mathbf{Z})$ will mix E and τ. However, only for the cases discussed above (such as when E represented an almost complex flat target space) one was able to identify an element of $Sp(4dg,\mathbf{Z})$ which would cause the change $Z_{class}(E,\tau) = Z_{class}(\tau,E)$. The importance of the role played by the $Sp(4dg,\mathbf{Z})$ is still unclear.

We have studied mathematical identities of the classical piece of the bosonic string partition functions. The formal interchange of the worldsheet with the target space does not apply in an obvious way to the quantum part of the partition function. If it so happens that a string theory contains only a finite number of target space particles, no worldsheet oscillators exist, and the mathematical identities may acquire some physical reality. In the next section we discuss one such case.

5.2 Worldsheet/Target-Space Duality of $N = 2$ Strings

Symplectic structures emerged in considering worldsheet/target-space duality in the case when the target space was flat. A similar structure was considered also in a case of non-flat backgrounds [241]. The setting is a theory with *local* $N = 2$ worldsheet supersymmetry [1]. Gauge fixing in such theories leads to the following ghost systems: one associated with general coordinate invariance which contributes -26 units to the central charge c, two associated with the two local supersymmetries, contributing $c = 22$, and one associated with a $U(1)$ gauge symmetry, contributing $c = -2$. All in all the matter system needs to contribute $c = 6$ to arrange for the cancellation of the total central charge. A system of four bosonic coordinates and four fermions has the required central charge. The reason that this four dimensional system has not been the

prime candidate for a string background is that $N = 2$ supersymmetry requires that the space has either a (4,0) or a (2,2) signature.

In the (2,2) signature the model is still rather interesting. We will just state a few of its features. The target space spectrum of this model consists of only a single massless scalar particle[46]. This particle is associated with a deformation of the Kähler structure of the target space. Furthermore, this model does not contain any oscillators: its partition function consists purely of what was called the classical part in the previous section. It is thus an interesting system for checking the physical realization of mathematical identities of the type discussed above.

It is suggested [241] to consider as the target space of this model a Ricci-flat Kähler manifold constructed by associating to a given Riemann surface a 4-dimensional symplectic space, called "the cotangent bundle of the Riemann surface." A cotangent space to a given manifold of dimension d is a $2d$-dimensional space. The extra d-dimensional variables can be considered as the conjugate momenta to the original coordinates, with all the extra structure a phase space entails. The target space itself has also moduli; it has an equal number of complex-deformation moduli and Kähler-class moduli. Thus, structurewise, there appears a threefold similarity in the case the "target space genus" equals the genus g of the worldsheet. The similarity is between the given worldsheet moduli, the complex-structure moduli of the target space, and the Kähler-class moduli of the target space. For a genus-one case the partition function is known and is triality symmetric under interchanges among all three moduli. It is conjectured [241] that the triality is a general property of the theory.

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[46] A similar claim is made for $c = 1$ systems coupled to 2-d gravity.

Table 1: Summary of Target Space Duality in Different Settings

Background	The moduli-generating group $\mathcal{G}$	The moduli space $\mathcal{M}$	The duality group $\mathcal{G}_d$	Remarks
d-dim toroidal backgrounds	$O(d,d,\mathbf{R})$	$\frac{O(d,d,\mathbf{R})}{O(d,\mathbf{R})\times O(d,\mathbf{R})}$	$O(d,d,\mathbf{Z})\otimes_S \mathbf{Z}_2$	$\mathbf{Z}_2$ is the worldsheet parity
$D=4$ toroidal heterotic	$O(6,22,\mathbf{R})$	$\frac{O(6,22,\mathbf{R})}{O(6,\mathbf{R})\times O(22,\mathbf{R})}$	$O(6,22,\mathbf{Z})\otimes_S \mathbf{Z}_2$	$N=4$ $D=4$ SUSY
$D=4$ $\mathbf{Z}_N\subset SU(2)$ orbifolds		$\mathcal{M}=\mathcal{K}\times\mathcal{Q}$ $\mathcal{K}$=Kähler manifold $\mathcal{Q}$= quaternionic manifold		untwisted sector
$\mathbf{Z}_2$	$O(2,2,\mathbf{R})\times O(4,4,\mathbf{R})$	$\frac{O(2,2,\mathbf{R})}{O(2,\mathbf{R})\times O(2,\mathbf{R})}\times\frac{O(4,4,\mathbf{R})}{O(4,\mathbf{R})\times O(4,\mathbf{R})}$	$O(2,2,\mathbf{Z})\times O(4,4,\mathbf{Z})$	$N=2$ $D=4$ SUSY.
$\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_6$	$O(2,2,\mathbf{R})\times O(2,4,\mathbf{R})$	$\frac{O(2,2,\mathbf{R})}{O(2,\mathbf{R})\times O(2,\mathbf{R})}\times\frac{O(2,4,\mathbf{R})}{O(2,\mathbf{R})\times O(4,\mathbf{R})}$	$O(2,2,\mathbf{Z})\times O(2,4,\mathbf{Z})$	Submoduli of $T^2\times\mathcal{K}_3$
$T^2\times\mathcal{K}_3$	$O(2,2,\mathbf{R})\times O(4,20,\mathbf{R})$	$\frac{O(2,2,\mathbf{R})}{O(2,\mathbf{R})\times O(2,\mathbf{R})}\times\frac{O(4,20,\mathbf{R})}{O(4,\mathbf{R})\times O(20,\mathbf{R})}$	$O(2,2,\mathbf{Z})\times O(4,20,\mathbf{Z})$	conjectured
$\mathbf{Z}_N\subset SU(3)$ orbifolds				untwisted sector
$\mathbf{Z}_3$	$SU(3,3,\mathbf{R})$	$\frac{SU(3,3,\mathbf{R})}{U(1)\times SU(3,\mathbf{R})\times SU(3,\mathbf{R})}$	$SU(3,3,\mathbf{Z})$	$N=1$ $D=4$ SUSY.
$\mathbf{Z}_6$	$SU(2,2,\mathbf{R})\times SU(1,1,\mathbf{R})$	$\frac{SU(2,2,\mathbf{R})}{U(1)\times SU(2,\mathbf{R})\times SU(2,\mathbf{R})}\times\frac{SU(1,1,\mathbf{R})}{U(1)}$	$SU(2,2,\mathbf{Z})\times SU(1,1,\mathbf{Z})$	
$\mathbf{Z}_7,\mathbf{Z}_8,\mathbf{Z}_{12}$	$SU(1,1,\mathbf{R})^3$	$\left(\frac{SU(1,1,\mathbf{R})}{U(1)}\right)^3$	$SU(1,1,\mathbf{Z})^3$	Submoduli of CY spaces
$\mathbf{Z}_4$	$SU(2,2,\mathbf{R})\times O(2,2,\mathbf{R})$	$\frac{SU(2,2,\mathbf{R})}{U(1)\times SU(2,\mathbf{R})\times SU(2,\mathbf{R})}\times\frac{O(2,2,\mathbf{R})}{O(2,\mathbf{R})\times O(2,\mathbf{R})}$	$SU(2,2,\mathbf{Z})\times O(2,2,\mathbf{Z})$	
$\mathbf{Z}_{6'},\mathbf{Z}_{8'},\mathbf{Z}_{12'}$	$SU(1,1,\mathbf{R})^2\times O(2,2,\mathbf{R})$	$\left(\frac{SU(1,1,\mathbf{R})}{U(1)}\right)^2\times\frac{O(2,2,\mathbf{R})}{O(2,\mathbf{R})\times O(2,\mathbf{R})}$	$SU(1,1,\mathbf{Z})\times O(2,2,\mathbf{Z})$	
Calabi-Yau spaces	it does not exist in general	special Kähler manifold, $\mathcal{M}=\mathcal{M}_{(1,1)}\times\mathcal{M}_{(2,1)}$, unknown in general	mirror symmetry and monodromy group, unknown in general	
Curved backgrounds with $U(1)^d$ symmetries	$O(d,d,\mathbf{R})$	$\frac{O(d,d,\mathbf{R})}{\mathcal{H}}$ $O(d,\mathbf{R})_{diag}\subseteq\mathcal{H}\subseteq O(d,\mathbf{R})\times O(d,\mathbf{R})$	$O(d,d,\mathbf{Z})$	axial-vector duality is in $\mathcal{G}_d$

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FIGURE CAPTIONS

1.A: A string cannot tell if it is moving on circle (a) or circle (b).

1.B: Every point in the shaded region corresponds to an allowed string background. Points (a),(b),(c) correspond to isolated single backgrounds.

1.C: The line describes the space of compactifications on a circle of radius R_c. Arrows connect physically equivalent points related by duality.

1.D: An orbicircle obtained from a circle by identifying X with $-X$.

1.E: From the point of view of a string, a circle (at a special radius) is equivalent to an orbifold (at another special radius).

1.F: String propagation on (a), the semi-infinite cigar, is equivalent to propagation on (b), the infinite trumpet.

2.A: Genus-g Riemann surface with cycles a, b.

2.B: The $\rho = \tau$ slice of the fundamental domain of 2-d toroidal backgrounds. The points (a) and (b) are the enhanced symmetry points $SU(2)^2$ and $SU(3)$, respectively. The shaded region takes into account also the $B \to -B$ symmetry.

3.A: The $c = 1$ moduli space. Arrows connect physically equivalent points related by duality. The self-dual point, $R_c = 1$, possesses an enhanced $SU(2)_L \times SU(2)_R$ affine symmetry. The self-dual point, $R_o = 1$, possesses an enhanced $U(1)_L \times U(1)_R$ affine symmetry; it is equivalent to $R_c = 2\sqrt{2}$.

4.A: The maximal extension of the black hole solution in Kruskal coordinates.

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Unity of Superstring Dualities

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ABSTRACT

The effective action for type II string theory compactified on a six torus is $N = 8$ supergravity, which is known to have an E_7 duality symmetry. We show that this is broken by quantum effects to a discrete subgroup, $E_7(\mathbb{Z})$, which contains both the T-duality group $O(6,6;\mathbb{Z})$ and the S-duality group $SL(2;\mathbb{Z})$. We present evidence for the conjecture that $E_7(\mathbb{Z})$ is an exact 'U-duality' symmetry of type II string theory. This conjecture requires certain extreme black hole states to be identified with massive modes of the fundamental string. The gauge bosons from the Ramond-Ramond sector couple not to string excitations but to solitons. We discuss similar issues in the context of toroidal string compactifications to other dimensions, compactifications of the type II string on $K_3 \times T^2$ and compactifications of eleven-dimensional supermembrane theory.

1. Introduction

String theory in a given background can be formulated in terms of a sum over world-sheet fields, (super-) moduli and topologies of a world-sheet sigma-model with the background spacetime as its target space. Different backgrounds may define the same quantum string theory, however, in which case they must be identified. The transformations between equivalent backgrounds generally define a discrete group and such discrete gauge symmetries are referred to as duality symmetries of the string theory. An example is T-duality, which relates spacetime geometries possessing a compact abelian isometry group (see [1] and references therein). The simplest case arises from compactification of the string theory on a circle since a circle of radius R defines the same two-dimensional quantum field theory, and hence the same string theory, as that on a circle of radius α'/R. T-dualities are non-perturbative in the sigma-model coupling constant α' but valid order by order in the string coupling constant g. Some string theories may have additional discrete symmetries which are perturbative in α' but non-perturbative in g. An example is the conjectured S-duality of the heterotic string compactified on a six-torus [2,3,4]. In this paper we investigate duality symmetries of the type II string compactified to four dimensions and present evidence for a new 'U-duality' symmetry which unifies the S and T dualities and mixes sigma-model and string coupling constants.

Consider a compactified string for which the internal space is an n-torus with constant metric g_{ij} and antisymmetric tensor b_{ij}. The low-energy effective field theory includes a spacetime sigma model whose target space is the moduli space $O(n,n)/\big[O(n)\times O(n)\big]$ of the torus, and the constants g_{ij} and b_{ij} are the expectation values of the n^2 scalar fields. There is a natural action of $O(n,n)$ on the moduli space. In general this takes one string theory into a different one but a discrete $O(n,n;\mathbb{Z})$ subgroup takes a given string theory into an equivalent one. This is the T-duality group of the toroidally-compactified string and the true moduli space of the string theory is the moduli space of the torus factored by the discrete T-duality

group. There is a generalization to Narain compactifications on the '(p,q)-torus' $T(p,q)$ for which the left-moving modes of the string are compactified on a p-torus and the right-moving ones on a q-torus [5]. In this case the moduli space is $O(p,q)/\big[O(p)\times O(q)\big]$ factored by the T-duality group $O(p,q;\mathbb{Z})$. The $T(6,22)$ case is relevant to the heterotic string compactified to four dimensions which has $O(6,22;\mathbb{Z})$ as its T-duality group. At a generic point in the moduli space the effective field theory is $N=4$ supergravity coupled to 22 abelian vector multiplets, giving a total of 28 abelian vector gauge fields [6] with gauge group $U(1)^{28}$. It follows from the compactness of the full gauge group for all 28 vector gauge fields that any electric or magnetic charges are quantized. The effective field theory has an $SL(2;\mathbb{R})\times O(6,22)$ invariance of the equations of motion which, due to the charge quantization and the fact that states carrying all types of charge can be found in the spectrum, is broken to the discrete subgroup $SL(2;\mathbb{Z})\times O(6,22;\mathbb{Z})$. The $O(6,22;\mathbb{Z})$ factor extends to the T-duality group of the full string theory. It has been conjectured that the $SL(2;\mathbb{Z})$ factor also extends to a symmetry of the full string theory [3]. This is the S-duality group of the heterotic string. It acts on the dilaton field Φ and the axion field ψ (obtained by dualizing the four-dimensional two-form gauge field $b_{\mu\nu}$ that couples to the string) via fractional linear transformations of the complex scalar $\psi+ie^{-\Phi}$ and on the abelian field strengths by a generalized electric-magnetic duality. One of the $SL(2;\mathbb{Z})$ transformations interchanges the electric and magnetic fields and, when $\psi=0$, takes Φ to $-\Phi$ which, since the expectation value of e^{Φ} can be identified with the string coupling constant g, takes g to $1/g$, and so interchanges strong and weak coupling.

Consider now the compactification of the type IIA or type IIB superstring to four dimensions on a six-torus. The low-energy effective field theory is $N=8$ supergravity [7], which has 28 abelian vector gauge fields and 70 scalar fields taking values in $E_{7(7)}/\big[SU(8)/Z_2\big]$. The equations of motion are invariant under the action of $E_{7(7)}$ [7], which contains $SL(2;\mathbb{R})\times O(6,6)$ as a maximal subgroup. We shall show that certain quantum mechanical effects break $E_{7(7)}$ to a discrete subgroup which we shall call $E_7(\mathbb{Z})$, and this implies a breaking of the maximal

$SL(2;\mathbb{R})\times O(6,6)$ subgroup to $SL(2;\mathbb{Z})\times O(6,6;\mathbb{Z})$. The $O(6,6;\mathbb{Z})$ factor extends to the full string theory as the T-duality group and it is natural to conjecture that the $SL(2;\mathbb{Z})$ factor also extends to the full string theory as an S-duality group. In fact, we shall present evidence for the much stronger conjecture that the full $E_7(\mathbb{Z})$ group (to be defined below) extends to the full string theory as a new unified duality group, which we call U-duality. U-duality acts on the abelian gauge fields through a generalized electromagnetic duality and on the 70 scalar fields, the constant parts of which can each be thought of as a coupling constant of the theory. The zero-mode of the dilaton is related to the string coupling g, while 21 of the scalar zero-modes are the moduli of the metric on the 6-torus, and the others parameterise the space of constant anti-symmetric tensor gauge fields on the six-torus. U-duality implies that all 70 coupling constants are on a similar footing despite the fact that the standard perturbative formulation of string theory assigns a special rôle to one of them.

Whereas T-duality is known to be an exact symmetry of string theory at each order in the string coupling constant g, the conjectured S-duality and U-duality are non-perturbative and so cannot be established within a perturbative formulation of string theory. However, it was pointed out in [4] in the context of the heterotic string that there are a number of quantities for which the tree level results are known to be, or believed to be, exact, allowing a check on S-duality by a perturbative, or semi-classical, calculation. We shall show that U-duality for the type II string passes the same tests.

First, for compactifications of the type II string that preserve at least $N = 4$ supersymmetry, the low energy effective field theory for the massless modes is a supergravity theory whose form is determined uniquely by its local symmetries and is therefore not changed by quantum corrections. Duality of the string theory therefore implies the duality invariance of the equations of motion of the supergravity theory. This prediction is easily checked because the symmetries of the $N \geq 4$ supergravity/matter theories have been known for some time. In particular, the equations of motion of $N = 8$ supergravity are U-duality invariant. Another quan-

tity that should be, and is, duality invariant is the set of values of electric and magnetic charges allowed by the Dirac-Schwinger-Zwanziger quantization condition. Third, the masses of states carrying electric or magnetic charges satisfy a Bogomolnyi bound which, for the compactifications considered here, is believed to be unrenormalized to arbitrary order in the string coupling constant. Duality invariance of the string theory requires this bound to be duality invariant. For soliton states the Bogomolnyi bound can be found from a classical bound on field configurations of the effective supergravity theory carrying electric or magnetic charges that generalizes the bound obtained in [8] for Maxwell-Einstein theory. We present this bound for $N = 8$ supergravity and show that it is U-duality invariant. Fourth, the spectrum of 'Bogomolnyi states' saturating the Bogomolnyi bound should also be duality invariant. These states include winding and momentum modes of the fundamental string and those found from quantization of solitons. We shall assume that soliton solutions of the type II string can be identified with those of its effective $N = 8$ supergravity theory, and these, as we shall see, include various types of extreme black hole*.

One of the main concerns of this paper will be the Bogomolnyi states of the type II string theory that break half the supersymmetry. As we shall see, the soliton states in this category arise from quantization of a particular class of extreme black hole solution of $N = 8$ supergravity. It is essentially automatic that all soliton states of the type II string fall into representations of the U-duality group because this is a symmetry of the equations of motion of which the solitons are solutions†. A similar argument can be made for solitons of the heterotic string; for example, extreme black hole solutions of the low-energy field theory corresponding to the heterotic string fit into $SL(2,\mathbb{Z})$ representations [10]. There are two points to bear in mind, however. First, a duality transformation not only produces new soliton solutions from old but also changes the vacuum, as the vacuum is parame-

$\star$ See [9] for a discussion of the interpretation of extreme black holes as solitons

$\dagger$ They also fall into supermultiplets because of the fermion zero modes in the presence of an extreme black hole [11]

terised by the scalar expectation values and these change under duality. We shall assume, as in [4], that the new soliton state in the new vacuum can be continued back to give a new soliton state in the old vacuum with duality transformed electric and magnetic charges; this is certainly possible at the level of solutions of the low energy effective action, since the extreme black hole solutions depend analytically on the scalar expectation values. Combining U-duality transformations with analytic continuations of the scalar field zero-modes in this way gives an $E_7(\mathbb{Z})$ invariance of the spectrum of soliton states in a given vacuum. (Note that whereas U-duality preserves masses, combining this with a scalar zero-mode continuation gives a transformation which changes masses and so is not an invariance of the Hamiltonian.) Second, the four-dimensional metrics of many extreme black hole solitons are only defined up to a conformal rescaling by the exponential of a scalar field function that vanishes at spatial infinity. While the 'Einstein' metric is duality invariant, other metrics in the same conformal equivalence class will not be. In general one should therefore think of duality as acting on conformal equivalence classes of metrics and the issue arises as to which metric within this class is the physically relevant one. As we shall see, for the solutions considered here each conformal class of metrics contains one that is (i) either completely regular or regular outside and on an event horizon and (ii) such that its spatial sections interpolate between topologically distinct vacua. The extreme black hole solutions corresponding to these metrics might reasonably be interpreted as solitons of the theory.

We now encounter an apparent contradiction with U-duality, and with S-duality, of the type II string theory because the fundamental string excitations include additional Bogomolnyi states which apparently cannot be assigned to duality multiplets containing solitons because the soliton multiplets are already complete. The only escape from this contradiction is to make the hypothesis that the fundamental string states have already been counted among the soliton states. In order for this to be possible there must be soliton states carrying exactly the same quantum numbers as the fundamental Bogomolnyi states. This is indeed

the case. The idea that particles with masses larger than the Planck mass, and hence a Compton wavelength less than their Schwarzschild radius, should be regarded as black holes is an old one [12,13], and it has recently been argued that Bogomolnyi states in the excitation spectrum of the heterotic string should be identified with extreme electrically-charged dilaton black holes [14,15]. For the heterotic string, approximate solutions of the low-energy effective action include extreme black holes and self-gravitating BPS monopoles [16,17], and it is believed that these correspond to Bogomolnyi solitons of the heterotic string [4]. Any magnetically charged soliton will have an electrically charged soliton partner generated by the action of the $\mathbb{Z}_2$ electromagnetic duality subgroup of S-duality. Now, if the full string theory is S-duality invariant, and this $\mathbb{Z}_2$ subgroup acts on an electrically charged fundamental string state to give a magnetically charged soliton, as argued for the heterotic string in [4], then this fundamental string state must be identified with the corresponding electrically charged soliton. We shall return to these points later but it is worth noting here that solitons of the low-energy effective $N = 4$ or $N = 8$ supergravity theory fit into representations of the $S \times T$ or U duality as these are symmetries of the supergravity equations of motion, and this is true *irrespective of whether the duality symmetry is actually a symmetry of the full heterotic or type II string theory.*

For compactifications of ten-dimensional string theories one expects solitons of the effective four-dimensional theory to have a ten-dimensional origin. For the type II string we are able to identify the four dimensional solitons that break half the supersymmetry of $N = 8$ supergravity as six-torus 'compactifications' of the extreme black p-branes of either IIA or IIB ten-dimensional supergravity [18,19,20,21,22]. We note that, in this context, the Bogomolnyi bound satisfied by these states can be seen to arise from the algebra of Noether charges of the effective world-volume action [23]. Remarkably, the solitonic states that are required to be identified with fundamental string states are precisely those which have their ten-dimensional origin in the string soliton or extreme black 1-brane solution, which couples to the same two-form gauge field as the fundamental string. This suggests

that we should identify the fundamental ten dimensional string with the solitonic string. This is consistent with a suggestion made in [24], for other reasons, that the four-dimensional heterotic string be identified with an axion string.

A similar analysis can be carried out for non-toroidal compactification. A particularly interesting example is compactification of the type II superstring on $K_3 \times T^2$ [25] for which the effective four-dimensional field theory turns out to be identical to the effective field theory of the T^6-compactified heterotic string, and in particular has the same $SL(2,\mathbb{Z}) \times O(6,22;\mathbb{Z})$ duality group. Furthermore, the spectrum of extreme black hole states is also the same. This raises the possibility that the two string theories might be *non-perturbatively* equivalent, even though they differ perturbatively. Such an equivalence would clearly have significant implications.

Finally we consider similar issues in the context of the 11-dimensional supermembrane [26]. This couples naturally to 11-dimensional supergravity [27] and hence to $N = 8$ supergravity after compactification on T^7 and to $N = 4$ supergravity coupled to 22 vector multiplets after compactification on $K_3 \times T^3$ [28]. At present it is not known how to make sense of a quantum supermembrane, so there is little understanding of what the massive excitations might be. However, some progress can be made using the methods sketched above for the string. We shall show that, if the elementary supermembrane is identified with the solitonic membrane solution [29] of 11-dimensional supergravity and account is taken of the solitonic fivebrane solution [30], the results of this analysis for the four-dimensional theory are exactly the same as those of the type II string.

2. Charge Quantization and the Bogomolnyi Bound

Consider the four-dimensional Lagrangian

$$L = \sqrt{-g}\left(\frac{1}{4}R - \tfrac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{4}m_{IJ}(\phi)F^{\mu\nu}F^J_{\mu\nu} - \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}a_{IJ}(\phi)F^I_{\mu\nu}F^J_{\rho\sigma}\right) \tag{2.1}$$

for space-time 4-metric $g_{\mu\nu}$, scalars ϕ^i taking values in a sigma-model target space $\mathcal{M}$ with metric $g_{ij}(\phi)$, and k abelian vector fields A^I_μ with field strengths $F^I_{\mu\nu}$. The scalar functions $m_{IJ} + ia_{IJ}$ are entries of a positive definite $k \times k$ Hermitian matrix. The bosonic sector of all supergravity theories without scalar potentials or non-abelian gauge fields can be put in this form. We shall be interested in those cases for which the equations of motion are invariant under some symmetry group G, which is necessarily a subgroup of $Sp(2k;\mathbb{R})$ [31] and an isometry group of $\mathcal{M}$. Of principal interest here are the special cases for which $\mathcal{M}$ is the homogeneous space G/H where H is the maximal compact subgroup of G. These cases include many supergravity theories, and all those with $N \geq 4$ supersymmetry. For $N = 4$ supergravity coupled to m vector multiplets $k = 6 + m$, $G = SL(2;\mathbb{R}) \times O(6,m)$ and $H = U(1) \times O(6) \times O(m)$. For $N = 8$ supergravity $k = 28$, $G = E_{7(7)}$ and $H = SU(8)$. For the 'exceptional' $N = 2$ supergravity [32], $k = 28$, $G = E_{7(-25)}$ and $H = E_6 \times U(1)$.

Defining

$$G_{\mu\nu I} = m_{IJ}\star F^J_{\mu\nu} + a_{IJ}F^J_{\mu\nu} \tag{2.2}$$

where $\star F^I_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma I}$, the A^I_μ field equations and Bianchi identities can be written in terms of the the $2k$-vector of two-forms

$$\mathcal{F} = \begin{pmatrix} F^I \\ G_I \end{pmatrix} \tag{2.3}$$

as simply $d\mathcal{F} = 0$. The group G acts on the scalars through isometries of $\mathcal{M}$ and on $\mathcal{F}$ as $\mathcal{F} \to \Lambda\mathcal{F}$ where $\Lambda \in G \subseteq Sp(2k;\mathbb{R})$ is a $2k \times 2k$ matrix preserving the

$2k \times 2k$ matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{2.4}$$

An alternative way to represent the G/H sigma model is in terms of a G-valued field $V(x)$ which transforms under rigid G-transformations by right multiplication and under local H transformations by left multiplication

$$V(x) \to h(x)V(x)\Lambda^{-1} \qquad h \in H \ , \ \Lambda \in G \tag{2.5}$$

The local H-invariance can be used to set $V \in G/H$. Note that $\bar{\mathcal{F}} \equiv V\mathcal{F}$ is G-invariant. In most cases of interest, the scalar coset space can be parameterised by the complex scalars $z_{IJ} = a_{IJ} + im_{IJ}$ which take values in a generalised upper-half-plane (m_{IJ} is positive definite) and the group G acts on z_{IJ} by fractional linear transformations. (This can be seen for $N = 8$ supergravity as follows. In the symmetric gauge [7], the coset is parameterised by a scalar y_{IJ} which transforms under fractional linear transformations under G. However, z_{IJ} is related to y_{IJ} by a fractional linear transformation, $z_{IJ} = i(\mathbb{1} + \bar{y})/(\mathbb{1} - \bar{y})$, so that z in turn transforms under G by fractional linear transformations. Similar results follow for $N < 8$ supergravities by truncation.)

We now define the charges

$$Q^I = \oint_\Sigma \star F^I \qquad p_I = \frac{1}{2\pi} \oint_\Sigma F^I \qquad q_I = \oint_\Sigma G_I \tag{2.6}$$

as integrals of two-forms over a two-sphere Σ at spatial infinity. The charges p^I and q_I are the magnetic charges and the Noether electric charges, respectively. The charges Q^I are the electric charges describing the $1/r^2$ fall-off of the radial components of the electric fields, F^I_{0r}, and incorporate the shift in the electric charge of a dyon due to non-zero expectation values of axion fields [33]. Indeed, if

the scalars ϕ^i tend to constant values $\bar{\phi}^i$ at spatial infinity, then

$$q_I = m_{IJ}(\bar{\phi})Q^J + a_{IJ}(\bar{\phi})p^J \ . \tag{2.7}$$

The charges (p^I, q_I) can be combined into a $2k$-vector

$$\mathcal{Z} = \oint_\Sigma \mathcal{F} = \begin{pmatrix} p^I \\ q_I \end{pmatrix} \tag{2.8}$$

from which it is clear that $\mathcal{Z} \to \Lambda\mathcal{Z}$ under G.

The Dirac-Schwinger-Zwanziger (DSZ) quantization condition (with $\hbar = 1$) for two dyons with charge vectors $\mathcal{Z}$ and $\mathcal{Z}'$ is

$$\mathcal{Z}^T\Omega\mathcal{Z}' \equiv p^I q'_I - p'^I q_I = \nu \tag{2.9}$$

for some integer ν. This quantization condition is manifestly G-invariant as $G \subseteq Sp(2k;\mathbb{R})$. However, it has implications for the quantum theory only if there exist both electric and magnetic charges. If, for example, there are no magnetic charges of one type then (2.9) places no constraint on the values of the corresponding electric charge. For the cases of interest to us here, we will show that there exist electric and magnetic charges of all types. We shall now proceed with our analysis of the general case assuming all types of charge exist and this, together with the quantization condition (2.9), implies that the Noether electric charges q_I lie in some lattice Γ and that the magnetic charges p^I lie in the dual lattice $\tilde{\Gamma}$. The group G is therefore broken to the discrete sub-group $G(\mathbb{Z})$ which has the property that a vector $\mathcal{Z} \in \Gamma \oplus \tilde{\Gamma}$ is taken to another vector in the same self-dual lattice. The subgroup of $Sp(2k)$ preserving the lattice is $Sp(2k;\mathbb{Z})$, so that the duality group is

$$G(\mathbb{Z}) = G \cap Sp(2k;\mathbb{Z}) \ . \tag{2.10}$$

For compact G, $G(\mathbb{Z})$ is a finite group while for non-compact G it is an infinite discrete group. If we choose a basis for the fields A^I so that the electric charges,

and hence the magnetic charges, are integers, then the lattice $\Gamma \oplus \tilde{\Gamma}$ is preserved by integer-valued matrices, so that $Sp(2k;\mathbb{Z})$ consists of integer-valued $2k \times 2k$ matrices preserving Ω, and $G(\mathbb{Z})$ is also represented by integer-valued $2k \times 2k$ matrices. Note that the group $G(\mathbb{Z})$ is independent of the geometry of the lattice, as any two lattices Γ, Γ' are related by a $GL(k,\mathbb{R})$ transformation, so that the corresponding discrete groups $G(\mathbb{Z})$, $G'(\mathbb{Z})$ are related by $GL(k,\mathbb{R})$ conjugation and so are isomorphic. For $N = 4$ supergravity coupled to 22 vector multiplets, $G(\mathbb{Z})$ is precisely the $S \times T$ duality group $SL(2;\mathbb{Z}) \times O(6,22;\mathbb{Z})$ of the toroidally-compactified heterotic string, which was observed previously to be the quantum symmetry group of this effective field theory [4]. For $N = 8$ supergravity $G(\mathbb{Z})$ is a discrete subgroup of $E_{7(7)}$ which we shall call $E_{7(7)}(\mathbb{Z})$ and abbreviate to $E_7(\mathbb{Z})$. It can be alternatively characterized as the subgroup of $Sp(56;\mathbb{Z})$ preserving the invariant quartic form of $E_{7(7)}$. From the explicit form of this invariant given in [7], it is straightforward to see that $E_7(\mathbb{Z})$ contains an $SL(8,\mathbb{Z})$ subgroup. We also have

$$E_7(\mathbb{Z}) \supset SL(2;\mathbb{Z}) \times O(6,6;\mathbb{Z}) \ , \tag{2.11}$$

so that $E_7(\mathbb{Z})$ contains the T-duality group of the toroidally-compactified type II string. The minimal extension of the S-duality conjecture for the heterotic string would be to suppose that the $SL(2;\mathbb{Z})$ factor extends to an S-duality group of the type II string, but it is natural to conjecture that the full discrete symmetry group is the much larger U-duality group $E_7(\mathbb{Z})$. $E_7(\mathbb{Z})$ is strictly larger than $SL(2;\mathbb{Z}) \times O(6,6;\mathbb{Z})$, as it also contains an $SL(8,\mathbb{Z})$ subgroup. In the next section we shall verify some predictions of U-duality for the spectrum of states saturating a gravitational version of the Bogomolnyi bound, i.e. the 'Bogomolnyi states'. However, before turning to the spectrum we should verify that the Bogomolnyi bound is itself U-duality invariant, since otherwise a U-duality transformation could take a state in the Bogomolnyi spectrum to one that is not in this spectrum.

Consider first the cases of pure $N = 4$ supergravity (without matter coupling) and $N = 8$ supergravity, for which $\mathcal{M} = G/H$ and $k = N(N-1)/2$. We define

$Y_{mn} = t^I_{mn}(\bar{q}_I + i\bar{p}^I)$ where $(\bar{p}^I, \bar{q}_I)$ are the components of the $2k$-vector $\bar{\mathcal{Z}} = \overline{V}\mathcal{Z}$ and $\overline{V}$ is the constant asymptotic value of the G-valued field V at spatial infinity. Here $m, n = 1, \ldots, N$ and $t^I_{mn} = -t^I_{nm}$ are the matrices generating the vector representation of $SO(N)$. The Y_{mn} appear in a global supersymmetry algebra as central charges [34,8,9,35] and this allows a derivation of a Bogomolnyi bound. The anti-symmetric complex $N \times N$ matrix Y_{mn} has $N/2$ complex skew eigen-values λ_a, $a = 1, \ldots, N/2$ and the bound on the ADM mass of the Maxwell-Einstein theory [8] can be generalized to [35]

$$M_{ADM} \geq max|\lambda_a| \tag{2.12}$$

Since $\mathcal{Z} \to \Lambda\mathcal{Z}$ and $\overline{V} \to \overline{V}\Lambda^{-1}$ under G, it follows that $\bar{\mathcal{Z}}$ and the λ_a are invariant under duality transformations, so the bound (2.12) is manifestly G-invariant. In the quantum theory this bound translates to a bound on the mass of the corresponding quantum state. Similar results apply to the case of $N = 4$ super-matter coupled to supergravity, with the difference that $t^I_{mn} = -t^I_{nm}$ are now certain scalar-field dependent matrices that 'convert' the $SO(6, m)$ index I to the $SO(6)$ composite index mn. Nevertheless, the charges λ_a remain duality-invariant.

If the moduli of all the eigenvalues are equal, $|\lambda_{a_1}| = |\lambda_{a_2}| = \ldots = |\lambda_{a_{N/2}}|$, then the bound (2.12) is equivalent to

$$M_{ADM} \geq \sqrt{\frac{2}{N}}\sqrt{|\bar{\mathcal{Z}}|^2} \tag{2.13}$$

where

$$|\bar{\mathcal{Z}}|^2 = \sum_a |\lambda_a|^2 = \frac{1}{2}\overline{Y}_{mn}\overline{Y}^{mn} = G_{IJ}\bar{p}^I\bar{p}^J + G_{IJ}\bar{q}_I\bar{q}_J \tag{2.14}$$

and $G_{IJ} = \frac{1}{2}t^I_{mn}t^J_{mn}$ is the identity matrix for pure supergravity, but is scalar dependent for the the matter-coupled $N = 4$ theory. However, in the general case of different eigenvalues, the bound (2.14) is strictly weaker than (2.12). If M_{ADM} is equal to the modulus of r of the eigenvalues λ_a, $M_{ADM} = |\lambda_{a_1}| = |\lambda_{a_2}| =$

$\ldots = |\lambda_{a_r}|$, for some r with $0 \leq r \leq N/2$, then the soliton with these charges spontaneously breaks the N original supersymmetries down to r supersymmetries, so that for solitons for which $r = N/2$ precisely half of the N supersymmetries are preserved and the bound (2.12) is equivalent to (2.14). The duality invariance of the bound (2.14) for $N = 4$ was previously pointed out in [3,4].

3. Spectrum of Bogomolnyi states

There are many massive states in the spectrum of toroidally-compactified string theories. The masses of those which do not couple to any of the $U(1)$ gauge fields cannot be calculated exactly. This is also true in general of those that do couple to one of the $U(1)$ gauge fields but the masses of such particles are bounded by their charges, as just described. It is believed that the masses of string states that saturate the bound are not renormalized for theories with at least $N = 4$ supersymmetry. If this is so then these masses can be computed exactly. Such 'Bogomolnyi states' arise in the theory from winding and Kaluza-Klein modes of the fundamental string, and from quantization of non-perturbative soliton solutions of the string theory. The latter include extreme black holes and, for the heterotic string, self-gravitating BPS monopoles.

For generic compactifications of both heterotic and type II strings there are 28 abelian gauge fields and so a possible 56 types of electric or magnetic charge. We shall identify solitons of the effective supergravity theory carrying each type of charge, thereby justifying the quantization condition on these charges. These solitons are various types of extreme black holes. Initially, at least, we shall be interested in solitons carrying only one type of charge, in which case we should consistently truncate the supergravity theory to one with only one non-zero field strength, F. The coefficients of the F^2 terms can then be expressed in terms of a scalar field σ and a pseudoscalar field ρ (which are two functions of the ϕ^i) such that the truncated field theory has an action of the form

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{4}R + \frac{1}{4}e^{-2a\sigma}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\rho F_{\mu\nu}\star F^{\mu\nu} + L(\sigma,\rho)\right) \qquad (3.1)$$

where $L(\sigma,\rho)$ is the lagrangian for a scalar sigma-model and a is a constant. One can choose $a \geq 0$ without loss of generality since a is changed to $-a$ by the field redefinition $\sigma \to -\sigma$. For every value of a the equations of motion of (3.1) admit extreme multi-black hole solutions [36], parameterised by the asymptotic values of σ, ρ, which are arbitrary integration constants. There is an intrinsic ambiguity in the metric of the $a \neq 0$ extreme black hole solutions because a new metric can be constructed from the canonical metric (appearing in the action (3.1)) by a conformal rescaling by a power of e^{σ}. The general metric in this conformal equivalence class will not have an interpretation as a 'soliton' in the sense for which the $a = 0$ extreme Reissner-Nordstrom (RN) black hole is a soliton. One feature that is generally expected from a soliton is that it interpolate between different vacua: in the RN case these are the Minkowski spacetime near spatial infinity and the Robinson-Bertotti vacuum down an infinite Einstein-Rosen 'throat'. If we require of the $a \neq 0$ extreme black holes that they have a similar property then one must rescale the canonical metric $d\hat{s}^2$ by $e^{2a\sigma}$, after which one finds, for vanishing asymptotic values of σ and ρ, the solution

$$\begin{aligned} ds^2 &= e^{2a\sigma} d\hat{s}^2 \\ &= -\Big(1 - \frac{(1+a^2)M}{r}\Big)^{\frac{2(1-a^2)}{(1+a^2)}} dt^2 + \Big(1 - \frac{(1+a^2)M}{r}\Big)^{-2} dr^2 + r^2 d\Omega_2^2 \qquad (3.2) \\ e^{-a\sigma} &= \Big(1 - \frac{(1+a^2)M}{r}\Big)^{\frac{a^2}{1+a^2}}, \qquad \rho = 0 \end{aligned}$$

where M is the ADM mass and $d\Omega_2^2$ is the metric on the unit 2-sphere. When $a = 1$ and σ is the dilaton field this rescaling of the canonical metric is exactly what is required to get the so-called 'string metric', so that the $a = 1$ black holes have a natural interpretation as string solitons. This might make it appear that the rescaling of the canonical metric by $e^{2a\sigma}$ is inappropriate to string theory when $a \neq 1$, but it must be remembered that the scalar field σ is not necessarily the dilaton but is, in general, a combination of the dilaton and modulus fields of the torus and gauge fields. Indeed, it was shown in [37] that for the $a = \sqrt{3}$ black holes this combination is such that the effective rescaling is just that of (3.2). For

any value of a this metric has an internal infinity as $r \to (1+a^2)M$ for constant t. For $a < 1$ the surface $r = (1+a^2)M$ is an event horizon, but this event horizon is regular only if $2(1-a^2)/(1+a^2)$ is an integer, which restricts the values of a less than unity to $a = 0$ or $a = 1/\sqrt{3}$. The $a = 0$ case is the extreme RN black hole for which the soliton interpretation is widely accepted. The significance of the $a = 1/\sqrt{3}$ case has been explained in [38]. For $a \geq 1$ the surface $r = (1+a^2)M$ is at infinite affine parameter along any geodesic so one might admit all values of $a \geq 1$. On the other hand, the relevance of geodesic completeness is not clear in this context so one might still wish to insist that $2(1-a^2)/(1+a^2)$ be an integer so that the null surface $r = (1+a^2)M$ is regular, in which case only the further values of $a = 1$ and $a = \sqrt{3}$ can be admitted. Curiously, the values

$$a = 0, \ \frac{1}{\sqrt{3}}, \ 1, \ \sqrt{3} \tag{3.3}$$

which we find in this way by demanding that the solution (3.2) is a *bone fide* soliton also arise from truncation of $N = 8$ supergravity. The possibility of the values $a = 0$ and $a = 1$ is guaranteed by the existence of consistent truncations of $N = 8$ supergravity to $N = 2$ and $N = 4$ supergravity respectively. The possibility of the values $a = \sqrt{3}$ and $a = \frac{1}{\sqrt{3}}$ is guaranteed by the existence of a consistent truncation of the maximal five-dimensional supergravity to simple five-dimensional supergravity since the subsequent reduction to four dimensions yields just these values.

Consider first the $a = 0$, electric and magnetic extreme RN black holes. Given any one such black hole with integral charge, an infinite number can be generated by acting with $G(\mathbb{Z})$, and these will include black holes carrying each of the 56 types of charge [35], and this is already sufficient to show that the continuous duality group $E_{7(7)}$ is broken to a discrete subgroup. These solutions break 3/4 of the supersymmetry in the $N = 4$ theories and 7/8 of the supersymmetry in the $N = 8$ case. For the remainder of the paper, we shall restrict ourselves to solitons which break half the supersymmetry, and the only extreme black hole

solutions of this type are those with $a = \sqrt{3}$. This follows from consideration of the implications of supersymmetry for the moduli space of multi-black hole solutions. This multi-soliton moduli space is the target space for an effective sigma model describing non-relativistic solitons [39]. This sigma model must have 8 supersymmetries for solitons of a four-dimensional $N = 4$ supergravity theory that break half the supersymmetry, and this implies that the moduli space is hyper-Kahler. Similarly, the moduli space for multi-solitons of $N = 8$ supergravity that break half the supersymmetry is the target space for a sigma model with 16 supersymmetries, and this implies that the moduli space is flat. However, the moduli space of multi-black hole solutions is flat if and only if $a = \sqrt{3}$ [40,41], so only these extreme black holes can be solutions of $N = 8$ supergravity that break half the supersymmetry. An alternative characterization of these extreme black holes is as 'compactifications' of the extreme black p-brane solitons of the ten-dimensional supergravity theory, which are known to break half the supersymmetry [42]. It follows that the moduli space of these solutions must be flat and what evidence there is [43] confirms this prediction. This ten-dimensional interpretation of the solitons discussed here will be left to the following section where it will also become clear that they carry combinations of all $28 + 28$ electric and magnetic charges associated with the 28 $U(1)$ gauge fields.

This moduli space argument shows, incidentally, that whereas the flatness of the moduli space for solitons that break half the supersymmetry is protected by supersymmetry for $N = 8$ supergravity, this is not so for $N = 4$ theories. There is then no reason to expect the moduli space metric of extreme black hole solitons of the exact heterotic string theory (to all orders in α' and g) to be flat. Indeed, the $a = \sqrt{3}$ extreme black holes, which have a flat moduli space, are only approximate solutions of the heterotic string and are expected to receive higher order corrections. Furthermore, if BPS-type monopoles were to occur in the type II string theory, a possibility that is suggested by the occurrence of non-abelian gauge groups in some versions of the compactified type II string [44], they would have to break more than half the supersymmetry as their moduli space is not flat. This is in accord with

the fact that the four-dimensional type II strings of ref. [44] have at most $N = 4$ supersymmetry, so that solitons of these theories saturating a Bogomolnyi bound would have less than $N = 4$ supersymmetry. This provides further justification for our assumption that the solitons of the toroidally compactified type II string that break half the $N = 8$ supersymmetry are those of the effective $N = 8$ supergravity theory.

The complete set of soliton solutions of a supergravity theory fill out multiplets of the duality group $G(\mathbb{Z})$, as mentioned in the introduction. We shall now explain this in more detail. Flat four-dimensional space-time with the scalar fields ϕ^i taking constant values, ϕ^i_0, is a vacuum solution of the supergravity theory parameterised by these constants. The duality group acts non-trivially on such vacua as it changes the ϕ^i_0. The solitons for which the scalar fields tend asymptotically to the values ϕ^i_0 provide the solitonic Bogomolnyi states about the vacuum state $|\phi_0>$. A $G(\mathbb{Z})$ transformation takes a Bogomolnyi state in this vacuum with charge vector $\mathcal{Z}$ to another Bogomolnyi state with charge vector $\mathcal{Z}'$ and equal mass but in a new vacuum $|\phi'_0>$. As in [4], it will be assumed that one can smoothly continue from ${\phi'}_0$ to ϕ_0 without encountering a phase transition, to obtain a state with the charge vector $\mathcal{Z}'$, but a different mass in general, about the original vacuum $|\phi_0>$. This assumption seems reasonable because the extreme black hole solutions depend analytically on the constants ϕ^i_0. We thus obtain a new Bogomolnyi soliton solution about the original vacuum but with a $G(\mathbb{Z})$ transformed charge vector. The spectrum of all the Bogomolnyi states obtained in this way is $G(\mathbb{Z})$ invariant by construction. In particular, the number of these Bogomolnyi states with charge vector $\mathcal{Z}$ will be the same as the number with charge vector $\mathcal{Z}'$ whenever $\mathcal{Z}$ is related to $\mathcal{Z}'$ by a $G(\mathbb{Z})$-transformation.

In addition to the Bogomolnyi states that arise from solitons, there are also the electrically charged Bogomolnyi states of the fundamental string. These states are purely perturbative and for the type II string they consist of the Kaluza-Klein (KK) and winding modes of the string. If they are also to fit into multiplets of the duality group they must have magnetically charged partners under duality, and

these should be non-perturbative, i.e. solitonic. The soliton duality multiplets are, however, already complete for the reason just given. In order to have duality of the string theory we must therefore identify the fundamental states with electrically charged solitonic states. We shall see in the next section how this identification must be made.

It might be thought that all electrically charged soliton states should have an equivalent description in terms of fundamental states. This is presumably true of the heterotic string since there are fundamental string states carrying each of the 28 types of electric charge and these are related by the T-duality group $O(6,22;\mathbb{Z})$. In contrast, the fundamental modes of the type II string carry only 12 of the possible 28 electric charges, because the 16 Ramond-Ramond (RR) $U(1)$ gauge fields couple to the string through their field strengths only. The 12 string-mode electric charges are related by the T-duality group $O(6,6;\mathbb{Z})$ of the type II string. It would be consistent with S and T-duality to suppose that there are no charged states coupling to the 16 (RR) gauge fields, but this would not be consistent with U-duality, as we now show.

Recall that an n-dimensional representation of G gives an action of G on $\mathbb{R}^n$ which restricts to an action of $G(\mathbb{Z})$ on the lattice $\mathbb{Z}^n$. For both the heterotic and type II strings, the charge vector $\mathcal{Z}$ transforms under G as a 56-dimensional representation. For the heterotic string, $G = SL(2;\mathbb{R}) \times O(6,22)$ and $\mathcal{Z}$ transforms according to its irreducible $(\mathbf{2},\mathbf{28})$ representation. This has the decomposition

$$(\mathbf{2},\mathbf{28}) \to (\mathbf{2},\mathbf{12}) + 16 \times (\mathbf{2},\mathbf{1}) \tag{3.4}$$

in terms of representations of $SL(2;\mathbb{R}) \times O(6,6)$. This is to be compared with the type II string for which $G = E_{7(7)}$ and $\mathcal{Z}$ transforms according to its irreducible **56** representation, which has the decomposition

$$\mathbf{56} \to (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32}) \tag{3.5}$$

under $SL(2;\mathbb{R}) \times O(6,6)$. In both cases there is a common sector corresponding to the $(\mathbf{2},\mathbf{12})$ representation of $SL(2;\mathbb{R}) \times O(6,6)$, plus an additional 32-dimensional

representation corresponding, for the heterotic string, to the charges for the additional $U(1)^{16}$ gauge group and, for the type II strings, to the charges for the Ramond-Ramond (RR) sector gauge fields. It is remarkable that the latter fit into the irreducible spinor representation of $O(6,6)$. These decompositions of the **56** representation of G induce corresponding decompositions of representations of $G(\mathbb{Z})$ into representations of $SL(2;\mathbb{Z}) \times O(6,6;\mathbb{Z})$ on the charge lattice $\mathbb{Z}^{56}$. In particular, *U-duality requires the 16 +16 electric and magnetic charges of the RR sector to exist and to transform irreducibly under the action of the T-duality group* $O(6,6;\mathbb{Z})$, and we conclude that all charges in the RR sector must be carried by solitons. We shall later confirm this.

4. p-Brane interpretation of Bogomolnyi solitons

We have seen that the solitons of toroidally compactified superstrings fit into representations of the duality group $G(\mathbb{Z})$. Our concern here will be to identify states that break half the supersymmetry and carry just one of the 56 types of electric or magnetic charge. We shall call such states for which the charge takes the minimum value 'elementary'; acting on these with the duality group $G(\mathbb{Z})$ will generate a lattice of charged states. Here we wish to show how the elementary solitons arise from extreme black p-brane solitons of the ten dimensional effective supergravity theory. These may be of electric or magnetic type. Electric p-brane solitons give electrically charged solitons of the four-dimensional dimensionally reduced field theory, while magnetic ones give magnetic monopoles, provided we use the form of the four-dimensional supergravity theory that comes directly from dimensional reduction without performing any duality transformations on the one-form gauge fields (although we convert two-form gauge fields to scalar fields in the usual way). If we had chosen a different dual form of action, the solutions would be the same, but some of the electric charges would be viewed as magnetic ones, and vice versa. This form of the action is manifestly invariant under T-duality: for the heterotic string, the action is the $O(6,22)$-invariant one given in [45], which is

related to the one of [46] by a duality transformation, and for the type II string, it is a new $O(6,6)$-invariant form of the $N = 8$ supergravity action which is related to the $SL(8,\mathbb{R})$-invariant Cremmer-Julia action [7] by a duality transformation.

An extreme p-brane soliton of the ten-dimensional low-energy field theory has a metric of the form [21]

$$ds^2 = A(r)[-dt^2 + dx^i dx^i] + B(r)dr^2 + r^2 d\Omega^2_{8-p} \tag{4.1}$$

where x^i $(i = 1, \ldots, p)$ are p flat Euclidean dimensions, $d\Omega^2_{8-p}$ is the metric on an $(8-p)$-sphere, r is a radial coordinate, t is a time coordinate and $A(r), B(r)$ are two radial functions that tend to unity as $r \to \infty$. These solitons couple either to an anti-symmetric tensor gauge field A_r of rank $r = 7-p$, in which case the p-brane is magnetically charged and $F = dA$ is proportional to the $(8-p)$-sphere volume form ϵ_{8-p}, or one of rank $r = p-3$, in which case the brane is electric and $\star F$ is proportional to ϵ_{8-p}. In some cases, the p-brane solutions will have corrections of higher order in α', but some of the solutions correspond to exact conformal field theories.

We shall be interested in four dimensional solitons obtained by 'compactifying' p-brane solitons on the six-torus. Compactification on T^p is straightforward since one has only to 'wrap' the p-brane around the p-torus, which is achieved by making the appropriate identifications of the x^i coordinates. If $p < 6$ a soliton in four dimensions can then be found by taking periodic arrays on $T^{(6-p)}$ and making a periodic identification*. For example [47], a five-brane can be wrapped around a five-torus in six ways giving rise to six types of five-dimensional soliton and these yield six types of black-hole solitons in four dimensions on taking periodic arrays. Similarly, to 'compactify' a 0-brane (i.e. a 10-dimensional black hole) on a six-torus one first introduces a 6-dimensional periodic array of such black holes and

$\star$ Alternatively, since the solution of extreme black p-branes always reduces to the solution of the Laplace equation in the transverse space, one has only to solve this equation on $\mathbb{R}^3 \times T^{(6-p)}$ instead of $\mathbb{R}^{(9-p)}$ to find solitons of the four-dimensional theory.

periodically identifies. Instead of wrapping all p dimensions of a p-brane to obtain a point-like 0-brane in 3+1 dimensions, one can wrap $p-q$ dimensions to obtain a q-brane soliton in 3+1 dimensions; however, in what follows we shall restrict ourselves to 0-brane solitons in 4 dimensions.

The bosonic sectors of the ten-dimensional effective field theories of the heterotic and type IIA and type IIB superstrings each include a metric, g_{MN}, antisymmetric tensor gauge field, b_{MN}, and a dilaton field Φ. We shall first discuss this common sector of all three theories and then turn to the additional sectors characteristic of each theory. We expect the solutions we describe to be exact solutions of the classical type II theory, and their masses to be unrenormalized in the quantum theory, but for the heterotic string they are only approximate solutions (to lowest order in α') of the low energy field theory.

Dimensional reduction of the common (g, b, Φ) sector on T^6 yields 6 Kaluza-Klein abelian gauge fields $(g_{\mu i} + \cdots)$ coming from g_{MN} and another 6 abelian gauge fields $(b_{\mu i} + \cdots)$ coming from b_{MN}. It is straightforward to identify the magnetically-charged solitons associated with the KK gauge fields. These are the KK monopoles [48], consisting of the product of a self-dual Taub-NUT instanton, with topology $\mathbb{R}^4$, with a 5-torus and a time-like $\mathbb{R}$. As this is the the product of a five-metric with a five-torus, this can also be viewed as a fivebrane solution of the ten-dimensional theory wrapped around a five-torus.† There are six types of KK monopoles in four dimensions, one for each of the six KK gauge fields, because the fivebrane can be wrapped around the six-torus in six different ways. As four-dimensional solutions the KK monopoles are extreme black holes with $a = \sqrt{3}$, as expected from the moduli space argument of the previous section. The elementary magnetically charged solitons associated with the $b_{\mu i}$ gauge fields can be identified with the six possible 'compactifications' of the extreme black fivebrane [20,19] of the ten dimensional (g, b, Φ) theory. We shall refer to these as abelian H-monopoles;

† For fixed r, t, the solution has topology $S^3 \times T^5$, and the S^3 can be regarded as a Hopf bundle of S^1 over S^2. Thus locally it is $S^2 \times T^6$, so that this solution might also be thought of as a twisted 6-brane.

they were first given in [49] and have been discussed further in [50,47,22]. It is straightforward to check directly that the KK monopoles and the H-monopoles are indeed related by T-duality, as expected [51,52]. Note that we have not included KK modes of the 5-brane, i.e. configurations in which the 5-brane has momentum in some of the toroidal directions, as these either lead to extended objects in four dimensions, or to localised solitons that carry more than one type of charge and so are not elementary.

The KK and abelian H monopoles have electric duals. These electrically charged solitons have their ten-dimensional origin in the extreme black string [18] of the (g, b, Φ) theory, which is dual [53, 54,55] to the extreme black five-brane. The 6 electric duals to the abelian H-monopoles are found by wrapping the solitonic string around the 6-torus, i.e. the 6 winding modes of the solitonic string. The electric duals of the KK monopoles come from Kaluza Klein modes of the 1-brane, i.e. configurations in which the solitonic string has momentum in the toroidal directions. They can be thought of as pp-waves travelling in the compactified directions [11]. These $6+6$ elementary electrically charged solitons couple to the $6+6$ KK and $b_{\mu i}$ gauge fields. They are in one to one correspondence with the KK (i.e. torus momentum modes) and winding states of the fundamental string which couple to the same 12 gauge fields. This allows us, in principle, to identify the fundamental string states as soliton states and, as explained in earlier sections, U-duality of string theory forces us to do so.

Before turning to solitons of the additional sector of each string theory, we shall first explain here why these field theory solitons are exact solutions of type II string theory. Type II string theory in a (g, b, Φ) background is described by a non-linear sigma-model with (1,1) world-sheet supersymmetry. The KK monopole background is described by a (4,4) supersymmetric sigma-model plus a free (1,1) supersymmetric field theory; this is conformally invariant [56] and so gives an exact classical solution of string theory. The pp-wave background is also an exact classical solution [57], so that the T-duals of these two solutions must be exact classical solutions too. In contrast, the heterotic string in a (g, b, Φ) background

is described by a (1,0) supersymmetric sigma-model, and at least some of the solutions described above only satisfy the field equations to lowest order in α'. In some cases, as we will describe later, these solutions can be modified to obtain exact classical heterotic string solutions. However, it is not known in general whether such backgrounds can be modified by higher order corrections to give an exact string solution.

We have now accounted for $12+12$ of the required $28+28$ types of charge of all three ten-dimensional superstring theories. We now consider how the additional $16 + 16$ charges arise in each of these three theories, starting with the type II string. It is known that, after toroidal compactification, the type IIA and type IIB string theories are equivalent [58] but it is instructive to consider both of them. In either case, we showed in the last section that U-duality requires that the missing $16+16$ types of charge transform as the irreducible spinor representation of the T-duality group. Since T-duality is a perturbative symmetry, if there were electrically charged states of this type in the fundamental string spectrum, there would also have to be magnetic ones. However, magnetic charges only occur in the soliton sector, so a prediction of U-duality is that the corresponding 16 electric charges are also to be found in the soliton sector and not, as one might have thought, in the elementary string spectrum. We shall confirm this.

First we consider the type IIA theory. The ten-dimensional bosonic massless fields are the (g, b, Φ) fields of the common sector plus a one-form gauge potential, A_M, and a three-form gauge potential, A_{MNP}. These extra fields appear in the RR sector but couple to the string through their field strengths *only*. Upon compactification to four dimensions, A_M gives one abelian gauge field A_μ and A_{MNP} gives 15 abelian gauge fields A_μ^{ij}. These also couple to the string through their field strengths only and so there are no elementary string excitations that are electrically-charged with respect to these 16 gauge fields, as expected. The solitonic p-brane solutions of the ten-dimensional field theory involving A_M or A_{MNP} and breaking only half the supersymmetry consist of a 0-brane, i.e. a (ten-dimensional) extreme black hole, a 2-brane (i.e. a membrane), a 4-brane and a

6-brane. The 0-brane and the 2-brane are of electric type. The 0-brane gives rise to an electrically-charged four-dimensional black hole in the toroidally-compactified theory by the procedure of taking periodic arrays of the ten-dimensional solution. The membrane gives a total of $\frac{6\times5}{2} = 15$ electric black holes in four dimensions after 'wrapping' it around two directions of the six-torus and then taking periodic arrays to construct a four-dimensional solution. Similarly, the magnetic type 4-brane and 6-brane can be wrapped around the six-torus (introducing periodic arrays where necessary) to give 15+1 magnetically-charged black holes in four dimensions. We have therefore found a total of 32 additional electric and magnetic charges. Combined with the previous 24 charges this gives a total of 56 elementary charged states carrying only one type of charge. From the low-energy field theory we know that these charges transform according to the **56** representation of E_7, and that acting on these elementary solitons with $E_7(\mathbb{Z})$ generates a 56-dimensional charge lattice. As anticipated, the extra 16+16 electric and magnetic charges are inert under S-duality but are mixed by the T-duality group $O(6,6;\mathbb{Z})$. In addition to the p-brane winding modes discussed above, there are also p-brane momentum modes; however, to give a 0-brane in 4 dimensions, the p-brane must wrap around the torus as well as having internal momentum, so that the resulting soliton would carry more than one type of charge and so would not be elementary; nevertheless, these solitons occur in the charge lattices generated by the elementary solitons.

A similar analysis can be made for the type IIB theory. In this case the extra massless bosonic fields in the ten dimensional effective field theory are a scalar, a two-form potential, A_{MN}, and a four-form potential $A^{(+)}_{MNPQ}$ with self-dual five-form field strength. As for the type IIA string theory, these gauge fields couple through their field strengths only and so, again, there are no string excitations carrying the new electric charges. In the solitonic sector of the ten-dimensional field theory there is a neutral 5-brane, a self-dual 3-brane and a string, in addition to the string and neutral fivebrane of the (g, b, Φ) sector. The new neutral fivebrane gives 6 magnetic charges in four dimensions, the self-dual 3-brane gives 10 electric and 10 magnetic charges and the new string gives six electric charges. Note that

the new solitonic string couples to the 16 $U(1)$ gauge fields coming from A_{MN} and $A^{(+)}_{MNPQ}$. These 16+16 charges couple to A_{MN}, while the fundamental string and the solitonic string of the common sector both couple to b_{MN}; thus it may be consistent to identify the fundamental and common sector solitonic strings, but the new solitonic string cannot be identified with either. As for the type IIA string all 56 charges generate the irreducible **56**-dimensional representation of $E_7(\mathbb{Z})$.

Finally, we turn to the heterotic string. We have seen that the common sector solutions of the low-energy effective supergravity theory include 12 KK and abelian H monopoles, and their 12 electric duals, and under T-duality these must have 16+16 electric and magnetic black hole partners coupling to the 16 remaining $U(1)$ gauge fields. These have a ten-dimensional interpretation as the 0-branes and 8-branes of $N = 1$ ten-dimensional supergravity coupled to 16 abelian vector multiplets [21] (which can be taken to be those of the $U(1)^{16}$ subgroup of $E_8 \times E_8$ or $SO(32)/\mathbb{Z}_2$).

In addition to these black hole solutions, there are also BPS monopole solutions of the heterotic string arising from wrapping heterotic or gauge five-branes around the six-torus [16]. The BPS monopoles are not solutions of the effective supergravity theory with abelian gauge group, but it has been argued (e.g. in [47]) that there should be modifications of these monopoles that are solutions of the abelian theory. The moduli spaces for multi-soliton solutions of BPS monopoles are hyper-Kahler [59] while those for extreme $a = \sqrt{3}$ black holes are flat [40,41], so that the black holes and BPS monopoles should not be related by duality. If the modified BPS monopoles also have a non-flat moduli space, then they too cannot be dual to black holes. However, it is also possible that they have a flat moduli space, and even that they are equivalent to black hole solutions. The modified BPS monopoles, if they exist, would have electric partners under S-duality which would be electric solitons. The magnetic partners under S-duality of electrically charged Bogomolnyi fundamental string states are expected to be magnetic monopole solitons, which might be either BPS-type solutions, or black holes (or both, if they are equivalent). In either case, the fundamental string states should be identified with

the electrically charged solitons related to the magnetic monopoles by S-duality.

Whereas the solutions of the type II string we have discussed are exact conformal field theories, the solutions of the heterotic string are only approximate low-energy solutions. However, some of these heterotic solutions have a compact holonomy group, and for these one can set the Yang-Mills connection equal to the spin-connection so that the sigma-model becomes one with (1,1) supersymmetry and the resulting background is an exact solution of string theory. Applying this to the five-brane gives the symmetric five-brane solution [50] and this can also be used to construct a 'symmetric KK monopole'. However, we do not know which of the other solutions of the low-energy effective theory can be corrected to give exact solutions, and the duality symmetry of non-abelian phases of the theory are not understood. Moreover the α' corrections to the four-dimensional supergravity action give a theory that is not S-duality invariant, but if string-loop corrections are also included, an S-duality invariant action should arise.

5. Toroidal compactification to other dimensions

In this section, we extend the previous discussion to consider the duality symmetries of type II and heterotic strings toroidally compactified to d dimensions. The resulting low-energy field theory is a d-dimensional supergravity theory which has a rigid 'duality' group G, which is a symmetry of the equations of motion, and in odd dimensions is in fact a symmetry of the action. In each case the massless scalar fields of the theory take values in G/H, where H is the maximal compact subgroup of G. G has an $O(10-d,10-d)$ subgroup for the type II string, and an $O(10-d,26-d)$ subgroup for the heterotic string. In either string theory, it is known that this subgroup is broken down to the discrete T-duality group, $O(10-d,10-d;\mathbb{Z})$ or $O(10-d,26-d;\mathbb{Z})$. It is natural to conjecture that the whole supergravity duality group G is broken down to a discrete subgroup $G(\mathbb{Z})$ (defined below) in the d-dimensional string theory. We have already seen that this occurs for $d=4$ and will argue that for $d>4$ the symmetry G is broken to a dis-

crete subgroup by a generalisation of the Dirac quantization condition. In tables 1 and 2, we list these groups for toroidally compactified superstring theories (at a generic point in the moduli space so that the gauge group is abelian).

Space-time Dimension d	Supergravity Duality Group G	String T-duality	Conjectured Full String Duality
$10A$	$SO(1,1)/\mathbb{Z}_2$	$\mathbb{1}$	$\mathbb{1}$
$10B$	$SL(2,\mathbb{R})$	$\mathbb{1}$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R})\times O(1,1)$	$\mathbb{Z}_2$	$SL(2,\mathbb{Z})\times\mathbb{Z}_2$
8	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$O(2,2;\mathbb{Z})$	$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$
7	$O(5,5)$	$O(3,3;\mathbb{Z})$	$O(5,5;\mathbb{Z})$
6	$SL(5,\mathbb{R})$	$O(4,4;\mathbb{Z})$	$SL(5,\mathbb{Z})$
5	$E_{6(6)}$	$O(5,5;\mathbb{Z})$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}$	$O(6,6;\mathbb{Z})$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}$	$O(7,7;\mathbb{Z})$	$E_{8(8)}(\mathbb{Z})$
2	$E_{9(9)}$	$O(8,8;\mathbb{Z})$	$E_{9(9)}(\mathbb{Z})$
1	$E_{10(10)}$	$O(9,9;\mathbb{Z})$	$E_{10(10)}(\mathbb{Z})$

Table 1 Duality symmetries for type II string compactified to d dimensions.

For the type II string, the supergravity duality groups G are given in [60, 61]*. The Lie algebra of $E_{9(9)}$ is the $E_{8(8)}$ Kac-Moody algebra, while the algebra corresponding to the E_{10} Dynkin diagram has been discussed in [61, 63]. The $d = 2$ duality symmetry contains the infinite-dimensional Geroch symmetry group of toroidally compactified general relativity. In $d = 9$, the conjectured duality group is a product of an $SL(2,\mathbb{Z})$ S-duality and a $\mathbb{Z}_2$ T-duality, while for $d < 8$ we conjecture a unified U-duality. For $d = 8$, the T-duality group $O(2,2;\mathbb{Z}) \sim [SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})]/\mathbb{Z}_2\times\mathbb{Z}_2$ is a subgroup of the conjectured duality. In $d = 10$, the type IIA string has $G = SO(1,1)/\mathbb{Z}_2$, while the type IIB has $G = SL(2,\mathbb{R})$, as indicated in the first two lines of the table. We shall abbreviate $E_{n(n)}(\mathbb{Z})$ to $E_n(\mathbb{Z})$ when no confusion can arise.

$\star$ They have also been discussed in the context of worldvolume actions of extended objects in supergravity backgrounds [62].

Space-time Dimension d	Supergravity Duality Group G	String T-duality	Conjectured Full String Duality
10	$O(16) \times SO(1,1)$	$O(16;\mathbb{Z})$	$O(16;\mathbb{Z}) \times \mathbb{Z}_2$
9	$O(1,17) \times SO(1,1)$	$O(1,17;\mathbb{Z})$	$O(1,17;\mathbb{Z}) \times \mathbb{Z}_2$
8	$O(2,18) \times SO(1,1)$	$O(2,18;\mathbb{Z})$	$O(2,18;\mathbb{Z}) \times \mathbb{Z}_2$
7	$O(3,19) \times SO(1,1)$	$O(3,19;\mathbb{Z})$	$O(3,19;\mathbb{Z}) \times \mathbb{Z}_2$
6	$O(4,20) \times SO(1,1)$	$O(4,20;\mathbb{Z})$	$O(4,20;\mathbb{Z}) \times \mathbb{Z}_2$
5	$O(5,21) \times SO(1,1)$	$O(5,21;\mathbb{Z})$	$O(5,21;\mathbb{Z}) \times \mathbb{Z}_2$
4	$O(6,22) \times SL(2,\mathbb{R})$	$O(6,22;\mathbb{Z})$	$O(6,22;\mathbb{Z}) \times SL(2,\mathbb{Z})$
3	$O(8,24)$	$O(7,23;\mathbb{Z})$	$O(8,24;\mathbb{Z})$
2	$O(8,24)^{(1)}$	$O(8,24;\mathbb{Z})$	$O(8,24)^{(1)}(\mathbb{Z})$

Table 2 Duality symmetries for heterotic string compactified to d dimensions.

For the heterotic string, the supergravity duality groups G for $d > 2$ can be found in articles collected in [64]. Pure $N = 4$ supergravity in $d = 4$ reduces to a theory with $G = SO(8,2)$ in $d = 3$ and to a theory with supergravity duality group given by the affine group $SO(8,2)^{(1)}$ in $d = 2$ [60]. Similar arguments suggest that the heterotic string should give a $d = 2$ supergravity theory with G given by the affine group $O(8,24)^{(1)}$ symmetry. The heterotic string is conjectured to have an $S \times T$ duality symmetry in $d \geq 4$ and a unified U-duality in $d \leq 3$. Sen conjectured an $O(8,24;\mathbb{Z})$ symmetry of $d = 3$ heterotic strings in [65]. The $d = 10$ supergravity theory has an $O(16)$ symmetry acting on the 16 abelian gauge fields which is broken to the finite group $O(16;\mathbb{Z})$; we refer to this as the T-duality symmetry of the ten-dimensional theory.

The supergravity symmetry group G in d dimensions doesn't act on the d-dimensional space-time and so survives dimensional reduction. Then G is necessarily a subgroup of the symmetry G' in $d' < d$ dimensions and dimensional reduction gives an embedding of G in G', and $G(\mathbb{Z})$ is a subgroup of $G'(\mathbb{Z})$. We use this embedding of G into the duality group in $d' = 4$ dimensions to define the duality group $G(\mathbb{Z})$ in $d > 4$ dimensions as $G \cap E_7(\mathbb{Z})$ for the type II string and as $G \cap [O(6,22;\mathbb{Z}) \times SL(2,\mathbb{Z})]$ for the heterotic string.

The symmetries in $d < 4$ dimensions can be understood using a type of argu-

ment first developed to describe the Geroch symmetry group of general relativity and used in [65] for $d = 3$ heterotic strings. The three-dimensional type II string can be regarded as a four-dimensional theory compactified on a circle and so is expected to have an $E_7(\mathbb{Z})$ symmetry. There would then be seven different $E_7(\mathbb{Z})$ symmetry groups of the three dimensional theory corresponding to each of the seven different ways of first compactifying from ten to four dimensions, and then from four to three. The seven $E_7(\mathbb{Z})$ groups and the $O(7,7;\mathbb{Z})$ T-duality group do not commute with each other and generate a discrete subgroup of E_8 which we define to be $E_8(\mathbb{Z})$. (Note that the corresponding Lie algebras, consisting of seven $E_{7(7)}$ algebras and an $O(7,7)$, generate the whole of the $E_{8(8)}$ Lie algebra.) Similarly, in $d = 2$ dimensions, there are eight $E_8(\mathbb{Z})$ symmetry groups and an $O(8,8;\mathbb{Z})$ T-duality group which generate $E_9(\mathbb{Z})$ as a discrete subgroup of $E_{9(9)}$, and in the heterotic string there are eight $O(8,24;\mathbb{Z})$ symmetry groups from three dimensions and an $O(8,24;\mathbb{Z})$ T-duality group which generate $O(8,24;\mathbb{Z})^{(1)}$ as a discrete subgroup of $O(8,24)^{(1)}$.

We now turn to the charge quantization condition and soliton spectrum in $d > 4$ dimensions. Consider first the example of type II string theory compactified to $d = 5$ dimensions. The low-energy theory is $d = 5$, $N = 8$ supergravity [66] which has 27 abelian vector gauge fields A^I_μ and an $E_{6(6)}$ rigid symmetry of the action. Recall that in five dimensions electric charge can be carried by particles or 0-brane solitons, while magnetic charge can be carried by strings or 1-brane solitons. The 27 types of electric charge q_I transform as a **27** of $E_{6(6)}$ while 27 types of magnetic charge p^I transform as a $\overline{\mathbf{27}}$. These charges satisfy the quantization condition $q_I p^I = integer$ [53,67] which is invariant under $E_{6(6)}$. As we shall see, all 54 types of charge occur and so the electric charges take values in a 27-dimensional lattice Λ and the magnetic ones take values in the dual lattice. This breaks the $E_{6(6)}$ symmetry down to the discrete subgroup which preserves the lattice. If the theory is now compactified to four dimensions, $E_{6(6)}$ survives as a subgroup of the $E_7(7)$ duality symmetry in $d = 4$ and the 27-dimensional lattice Λ survives as a sub-lattice of the 28-dimensional lattice of $d = 4$ electric charges (this will be

checked for the elementary charged solitons below). Thus the subgroup of $E_{6(6)}$ preserving Λ is $E_{6(6)} \cap E_7(\mathbb{Z})$, which is precisely the discrete group $E_6(\mathbb{Z})$ defined above.

The five-dimensional theory has a Bogomolnyi bound involving the electric and magnetic charges [11,17] which is saturated by Bogomolnyi solutions that do not break all the supersymmetries, and the masses and charges of these states are expected to be unrenormalised in the quantum theory. This bound is invariant under $E_{6(6)}$ and $E_6(\mathbb{Z})$ and so Bogomolnyi solitons automatically fit into $E_6(\mathbb{Z})$ representations.

The $d = 5$ elementary solitons of the type IIA theory carrying precisely one type of electric or magnetic charge and breaking half the supersymmetry can be identified in a similar manner to that used in $d = 4$. The 27 elementary electrically charged solitons, which are all extreme black hole solutions in $d = 5$, and the 27 magnetic ones, which are all extreme black strings, arise from $d = 10$ solutions as follows. The 5-brane wrapped around the 5-torus gives one electrically charged 0-brane and 5 magentically charged strings. The $d = 10$ solitonic string gives 5 electric black holes and 1 black string. The 0-brane gives one black hole, the 4-brane gives 10 black holes and 5 black strings, the 4-brane gives 5 black holes and 10 black strings and the 6-brane gives 1 black string. In addition, there are 5 electric black holes arising from pp-waves travelling in each of the 5 toroidal dimensions (these can also be viewed as momentum modes of the solitonic string), and their 5 magnetic duals, which are a magnetic string generalisation of the KK magnetic monopole. These are the solutions consisting of the product of a 4-torus with self-dual Taub-NUT and two-dimensional Minkowski space. The non-compact six-dimensional subspace gives rise to a 5-dimensional magnetic string in the same way that a 4-dimensional KK magnetic monopole originates from a five-dimensional solution. Note that these solutions can be thought of as wrapped 4-brane solutions. This gives the 27+27 elementary charges, as required.

On further compactification to $d = 4$, the 27 electric charges give 27 electri-

cally charged black holes in $d = 4$ and the magnetic strings give 27 magnetic black holes (together with 27 $d = 4$ black strings). There are two additional elementary charged states in $d = 4$, the pp-wave travelling in the fifth dimension and the KK monopole corresponding to the fifth dimension; these two solutions are uncharged from the five-dimensional point of view. This corresponds to the fact that the four-dimensional charges lie in a **56** of $E_{7(7)}$ and this decomposes into $E_{6(6)}$ representations as $\mathbf{56} \to \mathbf{27} + \overline{\mathbf{27}} + \mathbf{1} + \mathbf{1}$.

Similar arguments apply to other strings in $d \geq 4$ dimensions, where charge quantization effects break G to at most the string duality groups listed in the tables. In d dimensions there are electric point charges and magnetic $(d-4)$-brane solitons (which correspond to a subset of the four-dimensional black hole solitons on compactification) and the Dirac quantization of their charges [53,67] breaks the duality symmetry to the discrete subgroup $G(\mathbb{Z})$. There is a similar charge quantization condition on electric p-branes and magnetic $d - p - 4$ branes in d dimensions [53,67] which again break G to $G(\mathbb{Z})$. In each case, the Bogomolnyi solitons automatically fit into representations of the duality group (providing one can continue solutions from one vacuum to another as discussed in section 3).

For $d < 4$, it is not clear how to understand the breaking of G to $G(\mathbb{Z})$ directly in terms of d-dimensional quantum effects. Thus, while we have shown that for $d \geq 4$ the group G is broken to at most $G(\mathbb{Z})$, there is less evidence for our conjectures for $d < 4$, although we do know that a subgroup of G is broken to the discrete T-duality group, and that the solitons will fit into representations of $G(\mathbb{Z})$.

6. Compactification of Type II strings on $K_3 \times T^2$

The analysis of Bogomolnyi states can also be carried out for non-toroidal compactifications. An interesting example is compactification of the type II superstring on $K_3 \times T^2$ because while K_3 has no non-trivial one-cycles and hence no string winding modes it does have 22 non-trivial two-cycles around which a p-brane for $p > 1$ can wrap itself to produce a $(p-2)$-brane which will then produce monopole winding states on T^2 if $p < 5$ (taking periodic arrays where necessary). The effective four-dimensional supergravity can be found by the two stage process of compactification to six dimensions on K_3 [25], followed by a straightforward reduction on T^2. It is an $N = 4$ supergravity with an $SL(2;\mathbb{R}) \times O(6,22)$ symmetry and 28 $U(1)$ gauge fields, exactly as for the compactification of the heterotic string on T^6 at a generic point in the moduli space. In fact, the four-dimensional supergravity theories are identical because the coupling of $N = 4$ supergravity to k abelian vector multiplets is uniqely determined by the choice of gauge group [46]*. The analysis of section 2 again applies, with the result that the duality group is broken down to $SL(2;\mathbb{Z}) \times O(6,22;\mathbb{Z})$ by the charge quantization condition, and the Bogomolnyi bound is again duality invariant. The soliton spectrum then automatically fits into representations of $SL(2;\mathbb{Z}) \times O(6,22;\mathbb{Z})$ and the solitons correspond to precisely to the same extreme black holes as were discussed in section 4 for the heterotic string. However, the ten-dimensional origin of the elementary charged solutions is now different and we now discuss these.

Consider first the common (g, b, Φ) sector. Because K_3 has no isometries and no non-trivial one-cycles all KK modes and string winding modes arise from the T^2 compactification. This yields modes carrying 2+2 types of electric charge which couple to the 2+2 gauge fields from the metric and antisymmetric tensor. The corresponding magnetic charges are the KK monopoles and the H-monopoles. The

$\star$ Compactification of the heterotic string on $K_3 \times T^2$ leads to a four-dimensional effective field theory with only $N = 2$ supersymmetry, for which the masses of the Bogomolnyi states might be expected to receive quantum corrections, so we shall not discuss this case here.

latter can be interpreted as the winding modes on T^2 of the six-dimensional solitonic string found from 'wrapping' the neutral ten-dimensional fivebrane around the K_3 surface [68]. We have therefore identified the modes carrying just one type of the 4+4 electric and magnetic charges in this sector. As before, we do not consider modes arising from the ten-dimensional solitonic string on the grounds that these are not independent of the fundamental string modes already considered.

Consider now the type IIA string. The additional 24 vector gauge fields in the four dimensional effective field theory arise from the ten-dimensional RR gauge fields A_M and A_{MNP}. One of these vector gauge fields, A_μ, is the four-dimensional component of A_M. The remaining 23 come from expressing the ten-dimensional three-form A_{MNP} as the exterior product of a four-dimensional one-form gauge potential times each of the 22+1 harmonic two-forms of $K_3 \times T^2$. We must now find the charged Bogomolnyi states to which these fields couple. Again we consider states carrying only one type of charge. The ten-dimensional 0-brane and six-brane solitons associated with A_M yield, respectively, one electric and one magnetic four-dimensional black hole coupling to A_μ. The electric 2-brane and the magnetic 4-brane solitons in ten dimensions produce the Bogomolnyi states carrying the other 23+23 types of charge coupling to the other 23 gauge fields. Specifically, the 2-brane can be wrapped around the 22+1 non-trivial two-cycles of $K_3 \times T^2$ to produce 22+1 six-dimensional black holes of which one can then take periodic arrays to get 22+1 four-dimensional electric black holes. The four brane can be wrapped around the 22 homology two-cycles of K_3 to give 22 six-dimensional 2-branes, each of which can then be wrapped around T^2 to produce a four-dimensional magnetic black hole. Alternatively, the four-brane can be wrapped entirely around K_3 to give one six-dimensional black hole which then produces a further magnetic black hole in four dimensions on taking periodic arrays. We have now found a total of 24+24 additional electric and magnetic black holes. They each satisfy the Bogomolnyi bound, because the ten-dimensional p-brane solitons do, and they each carry just one type charge. Combining these with the 4+4 black holes from the (g, b, Φ) sector yields a total of 28+28 elementary electric and magnetic extreme black holes which

generate the $(\mathbf{2}, \mathbf{28})$ representation of $SL(2;\mathbb{Z}) \times O(6,22;\mathbb{Z})$.

Consider instead the type IIB superstring. The RR gauge fields are A_{MN} and $A^{(+)}_{MNPQ}$ which produce 2+22 four dimensional gauge fields upon compactification on $K_3 \times T^2$ [25]. These fields couple to the soliton states in four dimensions obtained by wrapping the extra solitonic string and fivebrane, and the self-dual three-brane, around the homology cycles of $K_3 \times T^2$, taking periodic arrays when necessary to get a four-dimensional soliton (alias extreme black hole). There are two homology one-cycles and two homology five-cycles so the extra solitonic string and fivebrane produce 2+2 four-dimensional electric and magnetic black holes. There are 44 three-cycles so the threebrane produces 44 four-dimensional solitons. Since the threebrane is self-dual 22 of these are electric and 22 magnetic. Again we have a total of 24+24 additional charges. Combining these with the 4+4 black holes from the (g, b, Φ) sector again yields a total of 28+28 elementary electric and magnetic extreme black holes which generate the $(\mathbf{2}, \mathbf{28})$ representation of $SL(2;\mathbb{Z}) \times O(6,22;\mathbb{Z})$.

Since the type II string compactified on $K_3 \times T^2$ and the generic toroidal compactification of the heterotic string have exactly the same four-dimensional low-energy field theory it is natural to conjecture that they might be equivalent string theories. If this is so then the Bogomolnyi states of the heterotic string discussed at the conclusion of the previous section would have a straightforward ten-dimensional interpretation after all. It would have some other remarkable consequences. For example, at special points of the heterotic string moduli space, there are extra massless fields and an enhanced (Yang-Mills) symmetry due to non-perturbative world-sheet effects (i.e. non-perturbative in α', but perturbative in g). If the compactified type II string is equivalent, it must have the same enhanced symmetry in vacua corresponding to the same points in the scalar field coset space. This presumably does not arise from non-perturbative world-sheet effects, so would have to come from non-perturbative stringy effects, or from Wilson lines and their p-brane generalisations. This would mean that the sigma-model coupling constant α' of the heterotic string becomes one of the stringy coupling constants of the

type II theory, as might have been expected from the fact that for the toridally compactified type II string, all coupling constants are on an equal footing and are mixed up under U-duality.

7. U-duality and the 11-dimensional supermembrane

As we have seen, the $E_7(\mathbb{Z})$ invariance of the spectrum of soliton states of $N = 8$ supergravity is an automatic consequence of the $E_7(\mathbb{Z})$ invariance of the equations of motion. The non-trivial features are, firstly, that these states have an interpretation in terms of ten-dimensional KK solitons and solitonic p-branes and, secondly, that if the ten-dimensional field theory is considered to be the effective field theory of the type II string theory then $E_7(\mathbb{Z})$ invariance requires an identification of the solitonic string with the fundamental string. $N = 8$ supergravity can also be obtained by dimensional reduction of 11-dimensional supergravity on T^7. We shall now show that the elementary soliton states of $N = 8$ supergravity also have an interpretation in terms of KK solitons and solitonic p-branes of 11-dimensional supergravity. There are 7 KK magnetic monopoles and 7 electric duals, which are pp-waves of 11-dimensional supergravity [69] travelling in the internal dimensions. The 11-dimensional solitonic p-branes are the electric membrane and the magnetic fivebrane. Each can be wrapped around the seven-torus to produce 21 four-dimensional solitons. Thus we have a total of 28 electric and 28 magnetic four-dimensional solitons each carrying one of the 56 types of charge which are a basis for the irreducible **56** repesentation of $E_7(\mathbb{Z})$. If we now wish to interpret 11-dimensional supergravity as an effective field theory of a fundamental $E_7(\mathbb{Z})$ supermembrane theory then we must identify the fundamental membrane with the solitonic one, just as we were forced to identify the fundamental string with the (appropriate) solitonic string.

Consider now the compactification to $\mathcal{M}_4$ on $K_3 \times T^3$ of eleven-dimensional supergravity [28]. The effective field theory is the same as that of the type II superstring compactified on $K_3 \times T^2$, i.e. an $N = 4$ supergravity with 28 $U(1)$

gauge fields and an $SL(2;\mathbb{Z}) \times O(6,22;\mathbb{Z})$ symmetry. The soliton spectrum is also the same. In the monopole sector we have, firstly, 3 KK monopoles from the T^3 factor and, secondly, a further 25 monopoles from wrapping the solitonic fivebrane around the 3+22 homology five-cycles of $K_3 \times T^2$. This gives a total of 28 monopoles. The 28 electrically charged solitons are the electric duals of these monopoles which can be understood in terms of the KK and winding modes of either the solitonic membrane or a fundamental membrane. The entire set of 56 states can be assigned to the (2,28) representation of $SL(2;\mathbb{R}) \times O(6,22)$, inducing a corresponding representation of $SL(2;\mathbb{Z}) \times O(6,22;\mathbb{Z})$.

These results are encouraging signs that it may be possible to define the quantum supermembrane theory entirely in terms of the solitonic membrane solution of eleven-dimensional supergravity. Alternatively, one can envisage a dual formulation in terms of a fundamental 11-dimensional the superfivebrane, in which case the solitonic fivebrane might be identified with a fundamental fivebrane. Of possible relevance in this connection is the fact that the membrane and fivebrane solitons have a very different global structure. Both have a degenerate Killing horizon but whereas the membrane horizon conceals a singularity in an interior region [70], much like the extreme Reissner Nordstrom solution of four-dimensional Maxwell-Einstein theory, the fivebrane is completely non-singular [38].

8. Comments

The equations of motion of four-dimensional effective supergravity theories of compactified superstring theories are invariant under a continuous duality group G that is broken by quantum effects to a discrete subgroup $G(\mathbb{Z})$. For the toroidally-compactified heterotic string at a generic point in the moduli space, and for the $K_3 \times T^2$-compactified type II superstrings, this group is the $S \times T$ duality group $SL(2;\mathbb{Z}) \times O(6,22;\mathbb{Z})$. For the toroidally-compactified type II superstrings it is the U duality group $E_7(\mathbb{Z})$ which contains the $S \times T$ duality group $SL(2;\mathbb{Z}) \times O(6,22;Z)$. Whereas T-duality is known to be an exact symmetry of string theory

at each order in the string coupling constant g, the conjectured S- and T- dualities are non-perturbative. We have provided evidence for U-duality of the type II string by considering those features of string theory that are expected to be given exactly by a semi-classical analysis although it should be emphasized that this evidence depends only on the form of the effective supergravity theory and would apply equally to any consistent quantum theory of gravity for which this is the effective low-energy action. Nevertheless, by supposing this consistent quantum theory to be string theory our arguments have led us to the remarkable conclusions that it is necessary to identify certain states of the string with extreme black holes, and the fundamental string with a solitonic string. We have also seen that the elementary Bogomolnyi states are extreme black hole solutions of the low-energy theory, and have shown how these arise from p-brane solitons of the ten-dimensional theory.

The zero modes of the scalar fields of the low energy field theory are all coupling constants of the string theory, so that $G(\mathbb{Z})$ symmetry relates different regimes in the perturbation theory in these coupling constants, interchanging strong and weak coupling and, in the case of U-duality, interchanging g with α', in the sense of mixing the quantum loop expansion in g with the sigma-model perturbation expansion in α' and the moduli of the compactification space. For the compactified type II superstring, any physical quantity (e.g. the S-matrix) can be expanded in terms of the 70 coupling constants associated with zero modes of the 70 scalars. In the world-sheet approach to string theory, one first integrates over the sigma-model degrees of freedom on a Riemann surface of fixed genus, obtaining a result parameterised by the sigma-model coupling constants, and then sums over genus. As U-duality mixes up all 70 coupling constants, the final result may be expected to depend on all 70 scalars in a symmetric way, even though the calculation was very asymmetrical and in particular picked out the dilaton to play a special role. This would hugely constrain the theory and the assumption of U-duality, together with $N = 8$ supersymmety, gives us a great deal of non-perturbative information, and might even enable us to solve the theory! This structure also suggests that there might be a new formulation of string theory which treats all the coupling

constants on an equal footing.

We have seen that there are duality symmetries in all dimensions $d \leq 10$ and it is interesting to ask whether the $d < 10$ symmetries can correspond to symmetries of the 10-dimensional theory. If e.g. the four-dimensional dualities correspond to symmetries of the full ten-dimensional theory, these symmetries must interchange the various p-brane solitons of the string-theory; such symmetries would probably have to be non-local. If the theories in $d < 4$ dimensions really do have the duality symmetries suggested in section 5, and if these have analogues in higher dimensions, this would have remarkable consequences for string theory. For example, the U-duality of the three-dimensional heterotic string includes transformations that would mix the string with 5-brane solitons in ten dimensions, and so would contain the transformations described as the 'duality of dualities' in [3]. For the type II string theory, this suggests that $E_{10}(\mathbb{Z})$ might be a discrete non-local symmetry of the ten-dimensional string theory!

One of the predictions of S-duality for the heterotic string is the presence of certain dyon bound states in the Bogomolnyi spectrum, which can be translated into to a prediction concerning harmonic forms on the multi-monopole moduli space [71]. It would be interesting to consider the corresponding predictions for the type II string. Recently, some strong-coupling evidence for S-duality of the heterotic string has been found by studying partition functions of certain topological field theories [72], and again it would be interesting to seek similar strong-coupling tests for U-duality.

Finally, since $N = 8$ supergravity and its soliton spectrum are U-duality invariant, many of the properties previously thought to be unique to string theory are in fact already properties of the effective supergravity theory once account is taken of all soliton solutions. Since only stable states can appear in an exact S-matrix it is possible that the only states of the exact toroidally compactified type II string theory are the Bogomolnyi states and that these are in one to one correspondence with soliton states of the supergravity theory. This would support

a previous suggestion [73] that a fundamental superstring theory might actually be equivalent to its effective field theory once solitons of the latter are taken into account. To pursue this further one would need to find higher spin (> 2) soliton states corresponding to the higher spin states of string theory. There seems to be no problem in principle with the existence of such higher-spin soliton states in $N = 8$ supergravity because the $a = 0$ and $a =$ 1extreme black holes must belong to massive supermultiplets with spins > 2 as they break more than half the supersymmetry. In this connection it is worth recalling the similarity of the mass/spin relation for Regge trajectories in four-dimensional string theory, $M^2 \propto \frac{J}{\alpha'}$, with that of the degenerate Kerr solutions of general relativity, $M^2 \propto \frac{J}{G}$.

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An Introduction to T-Duality in String Theory

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Abstract

In these lectures a general introduction to T-duality is given. In the abelian case the approaches of Buscher, and Roček and Verlinde are reviewed. Buscher's prescription for the dilaton transformation is recovered from a careful definition of the gauge integration measure. It is also shown how duality can be understood as a quite simple canonical transformation. Some aspects of non-abelian duality are also discussed, in particular what is known on relation to canonical transformations. Some implications of the existence of duality on the cosmological constant and the definition of distance in String Theory are also suggested.

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1 Introduction

Few words have been used with more different meanings than the word "duality". Even within the restricted framework of string theories, duality originally meant a symmetry between the s and the t-channels in strong interactions (coming from the demands in the S-matrix approach of the sixties of Regge behavior without fixed poles and analiticity, which were shown to imply the existence of an infinite number of resonances) [1]. Somewhat related ideas, also termed "duality", appear in the context of Conformal Field Theory (CFT) as simple consequences of locality and associativity of the operator product expansion (OPE) [2].

Duality symmetry plays an important rôle in Statistical Mechanics (for a review and references to the literature see for instance [3]), in particular in the analysis of the phase diagram of spin systems. It can also be understood as a way to show the equivalence between two apparently different theories. On a lattice system described by a Hamiltonian $H(g_i)$ with coupling constants g_i the duality transformation produces a new Hamiltonian $H^*(g_i^*)$ with coupling constants g_i^* on the dual lattice. In this way one can often relate the strong coupling regime of $H(g)$ with the weak coupling regime of $H^*(g^*)$. An important application was the determination of the exact temperature at which the phase transition of the two-dimensional Ising model takes place [4].

More recently, the word "duality" ("space-time duality") has been introduced in yet another sense. T-duality is a symmetry which relates physical properties corresponding to big spacetime radius with quantities corresponding to small radius. This will be our main theme in this review and from now on we will refer to it as just duality (a general reference is [5]). S-duality is a (conjectural) symmetry relating the strong coupling regime with the weak coupling one, a bold generalization of the original conjecture by Montonen and Olive [6]. Still more interesting (and speculative), there is a "duality of dualities": S-duality for strings corresponds to T-duality for fivebranes and conversely (see [7] for a general review). Another formally very similar property is β-duality, a property of the free energy of strings at finite temperature [8] which relates the high and the low temperature phases. For example, for the 10-dimensional heterotic string

$$F(\beta) = \frac{\pi^2}{\beta^2} F(\pi^2/\beta). \qquad (1.0.\ 1)$$

The physical interpretation of this symmetry is, however, somewhat uncertain due to the presence of the Hagedorn temperature.

In String Theory and Two-Dimensional Conformal Field Theory duality is an important tool to show the equivalence of different geometries and/or topologies and in determining some of the genuinely stringy implications on the structure of the low energy Quantum Field Theory limit. Duality symmetry was first described on the context of toroidal compactifications [9]. For the simplest case of a single compactified dimension of radius R, the entire physics of the interacting theory is left unchanged under the replacement $R \to \alpha'/R$ provided one also transforms the dilaton field $\phi \to \phi - \log(R/\sqrt{\alpha'})$ [10]. This simple case can be generalized to arbitrary toroidal compactifications described by constant metric g_{ij} and antisymmetric tensor b_{ij} [12]. The generalization of duality to this case becomes $(g+b) \to (g+b)^{-1}$ and $\phi \to \phi - \frac{1}{2}\log\det(g+b)$. In fact this transformation

is an element of an infinite order discrete symmetry group $O(d,d;Z)$ for d-dimensional toroidal compactifications [13, 14]. The symmetry was later extended to the case of non-flat conformal backgrounds in [16]. In Buscher's construction one starts with a manifold M with metric $g_{ij}, i,j = 0,\dots d-1$, antisymmetric tensor b_{ij} and dilaton field $\phi(x_i)$. One requires the metric to admit at least one continuous abelian isometry leaving invariant the σ-model action constructed out of (g,b,ϕ). Choosing an adapted coordinate system $(x^0,x^\alpha) = (\theta,x^\alpha), \alpha = 1,\dots d-1$ where the isometry acts by translations of θ, the change of g,b,ϕ is given by

$$\begin{aligned}
&\tilde{g}_{00} = 1/g_{00}, \qquad \tilde{g}_{0\alpha} = b_{0\alpha}/g_{00},\\
&\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - (g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta})/g_{00}\\
&\tilde{b}_{0\alpha} = g_{0\alpha}/g_{00},\\
&\tilde{b}_{\alpha\beta} = b_{\alpha\beta} - (g_{0\alpha}b_{0\beta} - g_{0\beta}b_{0\alpha})/g_{00},\\
&\tilde{\phi} = \phi - \frac{1}{2}\log g_{00}. \qquad (1.0.\ 2)
\end{aligned}$$

The final outcome is that for any continuous isometry of the metric which is a symmetry of the action one obtains the equivalence of two apparently very different non-linear σ-models. The transformation (1.0. 2) is referred to in the literature as abelian duality due to the abelian character of the isometry of the original σ-model. If n is the maximal number of commuting isometries, one gets a duality group of the form $O(n,n;Z)$ [18]. Duality symmetries are useful in determining important properties of the low-energy effective action, in particular in questions related to supersymmetry breaking and to the lifting of flat directions from the potential [15]. Although the transformation (1.0. 2) was originally obtained using a method apparently not compatible with general covariance, it is not difficult to modify the construction to eliminate this drawback [19]. A particularly useful interpretation of (1.0. 2) is in terms of the gauging of the isometry symmetry [17]. The duality transformation proceeds in two steps: i) First one gauges the isometry group, thus introducing some auxiliary gauge field variables A. The gauge field is required to be flat and this is implemented by adding a Lagrange multiplier term of the form χdA. It is naively clear that if we first perform the integral ovel χ, this provides a δ-function dA on the measure, implying that $A = dX$ is a pure gauge (we consider a spherical world sheet for simplicity). Fixing $X = 0$ the original model is recovered. ii) The second step consists of integrating first the gauge field A. Since there is no gauge kinetic term, the integration is gaussian, yielding a Lagrangian depending on the original variables and the auxiliary variable χ. After fixing the gauge the dual action follows. In [17] it was further shown that if one starts with a conformal field theory (CFT), conformal invariance is preserved by abelian duality. The proof was based on an analogy between the duality transformation and the GKO construction [32].

Of more recent history is the notion of non-abelian duality [22, 23, 24, 25], which has no analogue in Statistical Mechanics. The basic idea of [22], inspired in the treatment of abelian duality presented in [17], is to consider a conformal field theory with a non-abelian symmetry group G. In this case the gauge field variables A and the Lagrange multipliers live in the Lie algebra associated to G. The duality transformation proceeds in the two steps described above.

In the abelian case it is also possible to work out the mapping between some operators in the original and dual theories, as well as the global topology of the dual manifold [19]. Thus for G abelian we have a rather thorough understanding of the detailed local and global properties of duality. In the non-abelian case global information can only be extracted for σ-models with chiral currents [25]. For these models it is possible to perform a non-local change of variables in the Lagrange multiplier term such that the Lagrangian keeps its local expression and from it the global properties of the dual model can be worked out. The same construction does not work for general σ-models without chiral isometries.

Some interesting reviews on duality can be found in [5] and [20].

The organization of the lectures is as follows:

1. In section two we review the approaches of Buscher [16] and Roček and Verlinde [17] to abelian duality. We also exhibit the kind of information one can obtain with these formalisms. The approach of De la Ossa and Quevedo [22] to non-abelian duality is explained. Some comments are made concerning the global properties of the dual manifold [25].

2. In section three we show that for non-semisimple isometry groups a mixed gravitational-gauge anomaly may emerge in constructing the non-abelian dual. This explains in particular why the example considered in [24] violates conformal invariance to first order in $\alpha^{'}$.

3. In section four we study with some detail the transformation of the dilaton needed to preserve conformal invariance (to first order in $\alpha^{'}$) under duality.

4. In section five the problem of the behavior of the cosmological constant under duality is addressed. This study is motivated by the work in [36] where an explicit example in which the cosmological constant changes under a duality transformation is considered.

5. In section six we study the implications that duality has in the definition of a proper distance within String Theory. We consider particular families of correlators, manifestly duality invariant, and discuss the properties a distance based on them would have.

6. In section seven the canonical transformation approach to duality is studied. We shall be following [45]. In the abelian case the explicit generating functional producing Buscher's formulae is constructed. It is shown that all the information which can be obtained in the formulations above can be derived more easily this way. The general formulation of non-abelian duality as a canonical transformation is so far unknown. We review an example [47] where a non-abelian transformation in the $SU(2)$ principal chiral model is constructed as a canonical transformation of type I, the same type as for abelian duality.

7. Section eight contains a partial list of open problems.

2 Abelian and Non-Abelian Dualities

2.1 Abelian Duality

We start with a summary of Buscher's formulation [16]. Consider a non-linear σ-model defined on a d-dimensional manifold M:

$$S = \frac{1}{4\pi\alpha'}\int d^2\xi[\sqrt{h}h^{\mu\nu}g_{ij}\partial_\mu x^i\partial_\nu x^j + i\epsilon^{\mu\nu}b_{ij}\partial_\mu x^i\partial_\nu x^j + \alpha'\sqrt{h}R^{(2)}\phi(x)], \qquad (2.1.\ 1)$$

where g_{ij} is the target space metric, b_{ij} the torsion and ϕ the dilaton field, coupled to the two dimensional scalar curvature in the world sheet $R^{(2)}$. $h_{\mu\nu}$ is the world sheet metric and α' the inverse of the string tension. Let us assume that the σ-model has an abelian isometry represented by a translation in a coordinate θ in the target space. In the coordinates $\{\theta, x^\alpha\}$, $\alpha = 1, \ldots, d-1$, adapted to the isometry, the metric, torsion and dilaton fields are θ-independent. Then the original theory can be obtained from the following $d+1$-dimensional σ-model:

$$\begin{aligned} S_{d+1} = \frac{1}{4\pi\alpha'}\int d^2\xi[\sqrt{h}h^{\mu\nu}(g_{00}V_\mu V_\nu + 2g_{0\alpha}V_\mu\partial_\nu x^\alpha + g_{\alpha\beta}\partial_\mu x^\alpha\partial_\nu x^\beta) \\ +i\epsilon^{\mu\nu}(2b_{0\alpha}V_\mu\partial_\nu x^\alpha + b_{\alpha\beta}\partial_\mu x^\alpha\partial_\nu x^\beta) + 2i\epsilon^{\mu\nu}\tilde{\theta}\partial_\mu V_\nu + \alpha'\sqrt{h}R^{(2)}\phi(x)], \end{aligned} \qquad (2.1.\ 2)$$

where V is a 1-form defined on M and $\tilde{\theta}$ is an additional variable acting as a Lagrange multiplier. The equation of motion for $\tilde{\theta}$ implies $\epsilon^{\mu\nu}\partial_\mu V_\nu = 0$, which in topologically trivial world sheets forces $V_\mu = \partial_\mu\theta$, leading to the original theory. If instead we integrate over the V_μ-fields:

$$V_\mu = -\frac{1}{g_{00}}(g_{0\alpha}\partial_\mu x^\alpha + i\frac{\epsilon_\mu{}^\nu}{\sqrt{h}}(b_{0\alpha}\partial_\nu x^\alpha + \partial_\nu\tilde{\theta})), \qquad (2.1.\ 3)$$

we obtain the dual action:

$$\begin{aligned} \tilde{S} &= \frac{1}{4\pi\alpha'}\int d^2\xi[\sqrt{h}h^{\mu\nu}(\tilde{g}_{00}\partial_\mu\tilde{\theta}\partial_\nu\tilde{\theta} + 2\tilde{g}_{0\alpha}\partial_\mu\tilde{\theta}\partial_\nu x^\alpha + \tilde{g}_{\alpha\beta}\partial_\mu x^\alpha\partial_\nu x^\beta) \\ &\quad +i\epsilon^{\mu\nu}(2\tilde{b}_{0\alpha}\partial_\mu\tilde{\theta}\partial_\nu x^\alpha + \tilde{b}_{\alpha\beta}\partial_\mu x^\alpha\partial_\nu x^\beta) + \alpha'\sqrt{h}R^{(2)}\phi(x)], \end{aligned} \qquad (2.1.\ 4)$$

where:

$$\begin{aligned} \tilde{g}_{00} &= \frac{1}{g_{00}} \\ \tilde{g}_{0\alpha} &= \frac{b_{0\alpha}}{g_{00}}, \qquad \tilde{b}_{0\alpha} = \frac{g_{0\alpha}}{g_{00}} \\ \tilde{g}_{\alpha\beta} &= g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta}}{g_{00}} \\ \tilde{b}_{\alpha\beta} &= b_{\alpha\beta} - \frac{g_{0\alpha}b_{0\beta} - g_{0\beta}b_{0\alpha}}{g_{00}}. \end{aligned} \qquad (2.1.\ 5)$$

(2.1. 5) show that duality relates very different geometries. We will see that it may also lead to different topologies. The integration on V_μ produces a factor in the measure $\det g_{00}$ which conveniently regularized yields the shift of the dilaton:

$$\tilde{\phi} = \phi - \frac{1}{2}\log g_{00}. \qquad (2.1.\ 6)$$

The regularization prescription in order to find (2.1. 6) is fixed by requiring conformal invariance of the dual theory. In [16] the following definition was shown to yield the correct dilaton shift satisfying conformal invariance to first order in α':

$$\det A \equiv \frac{\det \Delta_A}{\det \Delta}, \tag{2.1. 7}$$

where $\Delta_A = -\frac{1}{\sqrt{h}}\partial_\mu(\sqrt{h}h^{\mu\nu}A\partial_\nu)$, $\Delta = -\frac{1}{\sqrt{h}}\partial_\mu(\sqrt{h}h^{\mu\nu}\partial_\nu)$. We will further justify this definition for the determinant in section four.

The σ-model defined by $(\tilde{g}, \tilde{b}, \tilde{\phi})$ is independent of the $\tilde{\theta}$ variable, hence the original model can be recovered by performing the duality transformation with respect to $\tilde{\theta}$ shifts.

This formalism has apparently some limitations:

1. It seems that general covariance is broken due to the choice of adapted coordinates needed to perform the duality transformation. This also obscures the issue of the global topology of the dual manifold, which is harder to describe if one works in local coordinates.

2. If the original theory has some isometries not commuting with the one used for duality they generically disappear as local symmetries in the dual model.

3. The original model is recovered from (2.1. 2) only in spherical world sheets. The monodromy of the V variable must be fixed by imposing the absence of modular anomalies. For that we need to know which are the orbits of the Killing vector.

4. When the Killing vector has fixed points, V_μ in (2.1. 3) is singular. In this case it could be much wiser to work with the $d+1$-dimensional action (2.1. 2).

5. What happens to the operator mapping from the above construction?.

6. What are the general properties of the non-abelian generalization?.

All these questions can be addressed with a different way of constructing the dual model. We will follow the work of Roček and Verlinde [17]. The formulation of Roček and Verlinde starts with the same σ-model (2.1. 1) with the abelian isometry represented by $\theta \to \theta + \epsilon$. The key point is to gauge the isometry by introducing some gauge fields A_μ transforming as $\delta A_\mu = -\partial_\mu \epsilon$. With a Lagrange multiplier term the gauge field strength is required to vanish, forcing the constraint that the gauge field is pure gauge. After gauge fixing the original model is then recovered.

Gauging the isometry in (2.1. 1) and adding the Lagrange multipliers term leads to:

$$\begin{aligned} S_{d+1} \;=\; & \frac{1}{4\pi\alpha'}\int d^2\xi[\sqrt{h}h^{\mu\nu}(g_{00}(\partial_\mu\theta + A_\mu)(\partial_\nu\theta + A_\nu) + 2g_{0\alpha}(\partial_\mu\theta + A_\mu)\partial_\nu x^\alpha \\ & + g_{\alpha\beta}\partial_\mu x^\alpha \partial_\nu x^\beta) + i\epsilon^{\mu\nu}(2b_{0\alpha}(\partial_\mu\theta + A_\mu)\partial_\nu x^\alpha + b_{\alpha\beta}\partial_\mu x^\alpha \partial_\nu x^\beta) + 2i\epsilon^{\mu\nu}\tilde{\theta}\partial_\mu A_\nu \\ & + \alpha'\sqrt{h}R^{(2)}\phi(x)]. \end{aligned} \tag{2.1. 8}$$

The dual theory is obtained integrating the A fields:

$$A_\mu = -\frac{1}{g_{00}}(g_{0\alpha}\partial_\mu x^\alpha + i\frac{\epsilon_\mu{}^\nu}{\sqrt{h}}(b_{0\alpha}\partial_\nu x^\alpha + \partial_\nu\tilde{\theta})), \tag{2.1. 9}$$

and fixing $\theta = 0$.

In [17] it is shown that the original and dual theories can be considered as the vectorial and axial cosets of a given higher dimensional theory with chiral currents in which the abelian symmetry group is gauged. This shows that conformal invariance is preserved by abelian duality to all orders in $\alpha^{'}$ since one can think of the initial and dual theories as two different functional integral representations of the same conformal field theory.

Within this approach the open questions enumerated above can be solved.

The procedure of gauging the isometry can be implemented in arbitrary coordinates [19]. If the original σ-model has a torsion term then Noether's procedure must be followed, as made explicit in [20]. Let us consider the following σ-model:

$$\begin{aligned} S &= \frac{1}{8\pi}\int g_{ij}\partial_\mu x^i \partial^\mu x^j + \frac{i}{8\pi}\int b_{ij} dx^i \wedge dx^j \\ &= \frac{1}{2\pi}\int d^2\xi (g_{ij}+b_{ij})\partial x^i \overline{\partial} x^j, \end{aligned} \qquad (2.1.\ 10)$$

where $\alpha^{'} = 2$. Let k^i be a Killing vector for the metric g:

$$\mathcal{L}_k g_{ij} = \nabla_i k_j + \nabla_j k_i = 0. \qquad (2.1.\ 11)$$

Invariance of S requires also

$$\mathcal{L}_k b = d\omega, \qquad \omega = i_k b - v, \qquad (2.1.\ 12)$$

where $(i_k b)_j \equiv k^i b_{ij}$ and v is a one-form such that $i_k H = -dv$ ($H = db$ locally). The associated conservation law is:

$$\partial \bar{J}_k + \bar{\partial} J_k = 0 \qquad (2.1.\ 13)$$

$$\begin{aligned} J_k &= (k - i_k b + \omega)_i \partial x^i = (k-v)_i \partial x^i \equiv (k-v)\cdot\partial x \\ \bar{J}_k &= (k + i_k b - \omega)_i \bar{\partial} x^i = (k+v)_i \bar{\partial} x^i \equiv (k+v)\cdot\bar{\partial} x. \end{aligned} \qquad (2.1.\ 14)$$

If we wish to gauge the isometry we introduce gauge fields $A, \bar{A}$, with $\delta_\epsilon A = -\partial\epsilon$, $\delta_\epsilon \bar{A} = -\bar{\partial}\epsilon$, and $\delta x^i = \epsilon k^i(x)$ now with ϵ a function on the world sheet. It can be shown [19] that the action:

$$S_{d+1} = \frac{1}{2\pi}\int d^2\xi [(g_{ij}+b_{ij})\partial x^i \bar{\partial} x^j + (J_k - \partial\chi)\bar{A} + (\bar{J}_k + \bar{\partial}\chi)A + k^2 A\bar{A}], \qquad (2.1.\ 15)$$

is invariant under:

$$\begin{aligned} \delta_\epsilon x^i &= \epsilon k^i(x) \qquad \delta_\epsilon \chi = -\epsilon k\cdot v \\ \delta_\epsilon A &= -\partial\epsilon \qquad \delta_\epsilon \bar{A} = -\bar{\partial}\epsilon. \end{aligned} \qquad (2.1.\ 16)$$

The Lagrange multiplier term forces the gauge field to be flat and at the same time cancels the anomalous variation of the Lagrangian. For a genus g world-sheet Σ_g and compact isometry orbits we may have large gauge transformations. We consider multivalued gauge functions:

$$\oint_\gamma d\epsilon = 2\pi n(\gamma) \qquad n(\gamma) \in Z, \qquad (2.1.\ 17)$$

where γ is a non-trivial homology cycle in Σ_g. Since we are dealing with abelian isometries it suffices to consider only the toroidal case $g = 1$. The variation of S_{d+1} is:

$$\begin{aligned} \delta S_{d+1} &= \frac{1}{2\pi}\int \left(\partial\chi\bar{\partial}\epsilon - \partial\epsilon\bar{\partial}\chi\right) = \frac{i}{4\pi}\int_T d\chi\wedge d\epsilon \\ &= \frac{i}{4\pi}\left(\oint_a d\chi \oint_b d\epsilon - \oint_a d\epsilon \oint_b d\chi\right) \end{aligned} \tag{2.1. 18}$$

where a and b are the two generators of the homology group of the torus T. Since ϵ is multivalued by $2\pi Z$, we learn from (2.1. 18) that χ is multivalued by $4\pi Z$:

$$\oint_\gamma d\chi = 4\pi m(\gamma) \qquad m(\gamma)\in Z. \tag{2.1. 19}$$

For a non-compact isometry $\delta S_{d+1} = 0$ and $d\chi$ may in general have real periods. The original theory is recovered integrating the Lagrange multiplier, which appears in the action in the form of a closed one form. In non trivial world sheets these one forms have exact and harmonic components. The χ-dependence in (2.1. 15) is:

$$S_\chi = -\frac{1}{2\pi}\int (d\chi_0 + \chi_h)\wedge A. \tag{2.1. 20}$$

Integrating by parts in the exact part and using Riemann's bilinear identity we obtain:

$$S_\chi = \frac{1}{2\pi}\int \chi_0\wedge dA - \frac{1}{2\pi}(\oint_a \chi_h \oint_b A - \oint_a A \oint_b \chi_h). \tag{2.1. 21}$$

Integration on χ_0 yields the constraint $dA = 0$ and integration on the harmonic components leads to:

$$\oint_a A = \oint_b A = 0. \tag{2.1. 22}$$

Both constraints imply that A must be an exact one form. Fixing the gauge the original theory is recovered. By construction S_{d+1} is general covariant, and therefore we have a clear idea of the d-dimensional geometrical interpretation of the model. Locally the dual manifold is equivalent to $(M/S^1)\times S^1$ (for compact isometries), where the quotient means that the gauge is fixed by dividing by the orbits of the isometry group. Generically we expect topology change as a consequence of duality. However the more delicate issue is whether the dual manifold $\tilde{M}$ is indeed a product or a twisted product (non-trivial bundle). It is also useful to notice that in the previous arguments the structure of $\pi_1(M)$ played no rôle. This rises some questions concerning the way the operators in both theories are mapped under duality [19]. The nature of the product relating the gauged original manifold and the Lagrange multipliers space turns out to be dictated by the gauge fixing procedure, in particular by Gribov problems. We use an example to labor this point. This is the $SU(2)$ principal chiral model, which represents a σ-model in S^3. The dual with respect to a fixed point free abelian isometry is locally $S^2\times S^1$. One knows that this also holds globally when performing the gauge fixing. This reveals that the dual manifold is $S^2\times S^1$ and not a squeezed S^3 (for details see [19]).

The interest of working with the $d+1$-dimensional theory is that the possible singularities in the dual theory due to the existence of fixed points do not emerge. However if

we are interested in the explicit form of the dual σ-model we have to eliminate the gauge field A. Integrating on A in (2.1. 15), the dual model $(\tilde{g}, \tilde{b}, \tilde{\phi})$ reads:

$$\begin{aligned}
\tilde{g}_{00} &= \frac{1}{k^2} \\
\tilde{g}_{0\alpha} &= \frac{v_\alpha}{k^2}, \qquad \tilde{b}_{0\alpha} = \frac{k_\alpha}{k^2} \\
\tilde{g}_{\alpha\beta} &= g_{\alpha\beta} - \frac{k_\alpha k_\beta - v_\alpha v_\beta}{k^2} \\
\tilde{b}_{\alpha\beta} &= b_{\alpha\beta} - \frac{k_\alpha v_\beta - k_\beta v_\alpha}{k^2} \\
\tilde{\phi} &= \phi - \frac{1}{2}\log k^2.
\end{aligned} \tag{2.1. 23}$$

Going to adapted coordinates and fixing the gauge we recover Buscher's formulae since $k^2 = g_{00}$, $v_\alpha = b_{0\alpha}$, $k_\alpha = g_{0\alpha}$. However this choice of coordinate system unables us to obtain global information about the dual manifold.

The explicit operator mapping can be constructed [19]. The duals to vertex operators $V_p = \exp ip\theta$, which are momenta operators in the direction of the isometry, are non-local operators which can be interpreted as winding operators only for flat compact isometries. Thus, the description of duality in toroidal compatifications as the symmetry exchanging momenta and windings is modified. In particular the structure of $\pi_1(M)$ turns out not to be important. The winding operators are associated to compact isometry orbits in the target space manifold and not to homologically non-trivial cycles as is usually interpreted for toroidal compactifications.

The extension to non-abelian isometry groups is easily done in this formalism. The details are worked out in the next section.

2.2 Non-Abelian Duality

The same procedure à la Roček and Verlinde was generalized in [22] to construct the dual with respect to a given no-abelian isometry group G. The gauge fields take values in the Lie algebra associated to the isometry group and they transform under gauge transformations $x^m \to g^m{}_n x^n$, $m, n = 1, \ldots, N$, where $g \in G$, as $A \to g(A + \partial)g^{-1}$. The isometry is gauged by introducing covariant derivatives[2]:

$$\partial x^m \to Dx^m = \partial x^m + A^\alpha (T_\alpha)^m{}_n x^n, \tag{2.2. 1}$$

where T_α is a N-dimensional representation for the α generator of the Lie algebra of G. The flatness of the gauge fields is imposed by the term:

$$\int Tr(\chi F), \tag{2.2. 2}$$

with $F = \partial\bar{A} - \bar{\partial}A + [A, \bar{A}]$. The χ-fields take values in the Lie algebra associated to G and transform in the adjoint representation to preserve gauge invariance. Integration on

[2]Note that this way of gauging a continuous global isometry is only valid for certain σ-models and isometry groups [26].

χ fixes $F=0$ in semisimple groups, then A is pure gauge (in spherical world sheets) and after gauge fixing we recover the original model. As before the dual model is obtained integrating on A and then fixing the gauge. For non-semisimple groups the Lagrange multipliers term must be introduced in a different way since the Cartan-Killing metric is degenerate and the integration on χ does not imply that all the F-components are zero. In this case the χ-fields must be taken in the basis dual to T_α and they transform in the coadjoint representation.

We can write the gauged σ-model action as:

$$\begin{aligned} S_{\text{gauge}} &= \frac{1}{2\pi}\int d^2z[Q_{mn}Dx^m\bar{D}x^n + Q_{m\mu}Dx^m\bar{\partial}x^\mu + Q_{\mu n}\partial x^\mu\bar{D}x^n + Q_{\mu\nu}\partial x^\mu\bar{\partial}x^\nu \\ &\quad + Tr(\chi F) + \frac{1}{2}R^{(2)}\phi], \end{aligned} \tag{2.2. 3}$$

where $Q=g+b$, latin indices are associated to coordinates adapted to the non-abelian isometry and greek indices to inert coordinates. We can write (2.2. 3) as:

$$S_{\text{gauge}} = S[x] + \frac{1}{2\pi}\int d^2z[\bar{A}^\alpha f_{\alpha\beta}A^\beta + \bar{h}_\alpha A^\alpha + h_\alpha\bar{A}^\alpha + \frac{1}{2}R^{(2)}\phi], \tag{2.2. 4}$$

with:

$$\begin{aligned} h_\alpha &= (Q_{mn}\partial x^m + Q_{\mu n}\partial x^\mu)(T_\alpha)^n{}_q x^q - \partial\chi^\alpha T_R\eta_{\alpha\alpha} \\ \bar{h}_\alpha &= (Q_{n\mu}\bar{\partial}x^\mu + Q_{nm}\bar{\partial}x^m)(T_\alpha)^n{}_q x^q + \bar{\partial}\chi^\alpha T_R\eta_{\alpha\alpha} \\ f_{\alpha\beta} &= Q_{mn}(T_\beta)^m{}_r(T_\alpha)^n{}_p x^r x^p + C_{\beta\alpha}{}^\gamma\chi^\gamma T_R\eta_{\gamma\gamma}, \end{aligned} \tag{2.2. 5}$$

where $[T_\alpha,T_\beta]=C_{\alpha\beta}{}^\gamma T_\gamma$ and $Tr(T_\alpha T_\beta)=T_R\eta_{\alpha\beta}$ ($Tr(T'_\alpha T_\beta)=T_R\eta_{\alpha\beta}$ if the group is not semisimple).

Integrating $A,\bar{A}$:

$$\tilde{S} = S[x] + \frac{1}{2\pi}\int d^2z[\bar{h}_\alpha(f^{\alpha\beta})^{-1}h_\beta + \frac{1}{2}R^{(2)}\tilde{\phi}], \tag{2.2. 6}$$

where $\tilde{\phi}$ is given by:

$$\tilde{\phi} = \phi - \frac{1}{2}\log(\det f), \tag{2.2. 7}$$

after regularizing the factor $\det f$ coming from the measure as in previous section. In all the examples considered the dual model with this dilaton satifies the conformal invariance conditions to first order in α', but a general proof analogous to that of Buscher in the abelian case is lacking.

The construction above seems to be a straightforward extension of abelian duality. However this is not so. Non-abelian duality is quite different from abelian duality, as it is clearly manifested in the context of Statistical Mechanics. In this context duality transformations are applied to models defined on a lattice L with physical variables taking values on some abelian group G. The duality transformation takes us from the triplet $(L,G,S[g])$, where $S[g]$ is the action depending on some coupling constants labelled collectively by g to a model $(L^*,G^*,S^*[g^*])$ on the dual lattice L^* with variables taking values on the dual group G^* and with some well-defined action $S^*[g^*]$. For abelian groups, G^*

is the representation ring, itself a group, and when we apply the duality transformation once again we obtain the original model. As soon as the group is non-abelian the previous construction breaks down because the representation ring of G is not a group [3]. In particular the non-abelian duality transformation cannot be performed again to obtain the model we started with. In the context of String Theory the major problems in stating non-abelian duality as an exact symmetry come when trying to extend it to non-trivial world sheets and when performing the operator mapping (a detailed explanation on this can be found in [19]). With the usual Lagrange multipliers variables is not possible to extract global information. In σ-models with chiral currents a non-local change of variables in the Lagrange multipliers term can be done such that the dual Lagrangian is local. In these variables the dual theory can be shown to be the product of the coset of the original manifold by the isometry group M/G and the WZW model of group G [25]. This applies in particular to the case of abelian groups, in agreement with the results on abelian duality. The dual variables introduced in [25] are the base of the non-abelian bosonization studied in [27]. As in the abelian case the more delicate issue is to know what kind of a product it is. This can be worked out in the case of WZW models [49]. The partition function at genus one of a WZW model with group G is not a product of any modular invariant partition function for the G/H coset theory and one of the H WZW-model (now H is the gauged isometry group). The Kac-Moody characters of the G_k WZW-model have a well-defined decomposition in terms of products of G/H and H_k characters. This implies that the product is a twisted product. In fact in this case the explicit integration on the A-fields can be made and the result is that the model is self-dual.

For non-abelian isometry groups certain anomalies can arise when performing the non-abelian dual construction [25, 28]. When one analyzes carefully the measure of integration over the gauge fields and its dependence on the world sheet metric, one encounters a mixed gauge and gravitational anomaly [29] when any generator of the isometry group in the adjoint representation has a non-vanishing trace. This can only happen for non-semisimple groups. This mixed anomaly generates a contribution to the trace anomaly which cannot be absorbed in a dilaton shift and imposes a mild anomaly cancellation condition for the consistency of non-abelian duality. We treat this point in the next section.

3 Mixed Anomalies and Effective Actions

Since we are interested in conformal invariance, we introduce an arbitrary metric $h_{\alpha\beta}$ on the world sheet and compute the contribution to the trace anomaly of the auxiliary gauge fields $A^a_\pm$. If for simplicity we work on genus zero surfaces, the most straightforward way to compute the dependence of the effective action on the world sheet metric is to first parametrize $A_\pm$ as:

$$A_+ = L^{-1}\partial_+ L, \;\; A_- = R^{-1}\partial_- R, \tag{3.0. 1}$$

for L, R group elements. We can think of $x^\pm$ as light-cone variables or as complex coordinates, and they depend on the metric being used. In changing variables from $A_\pm$ to (L, R) we encounter jacobians:

$$\mathcal{D}A_+\mathcal{D}A_- = \mathcal{D}L\,\mathcal{D}R\,\det(D_+(A_+)D_-(A_-)) \tag{3.0. 2}$$

with $A_\pm$ given by (3.0. 1) (we take $A_\pm$ as antihermitian matrices). We can write the determinants in (3.0. 2) in terms of a pair of (b,c)-systems (b_{+a}, c^a), $(b_{-a}, \tilde{c}^a)$. $c, \tilde{c}$ are 0-forms transforming in the adjoint representation of the group. For arbitrary groups $b_\pm$ transform in the coadjoint representation. The determinants in (3.0. 2) can be exponentiated in terms of the (b,c)- systems with an action:

$$S[b_\pm, c, \tilde{c}] = \frac{i}{\pi}\int (b_+ D_-(A)c + b_- D_+(A)\tilde{c}), \tag{3.0. 3}$$

which is formally conformal invariant. The variation of S with respect to the metric is given by the energy-momentum tensor $T_{\pm\pm}$. We can ignore momentarily that $A_\pm$ are given by (3.0. 1) and work with arbitrary gauge fields. We can compute the dependence of the effective action for (3.0. 3) on the metric $h_{\alpha\beta}$ and the gauge field using Feynman graphs. Expanding about the flat metric, and using the methods in [29], the first diagrams contributing to the effective action are

$h_{--}(h_{++})$ couples to $T_{++}(T_{--})$, and $A_-(A_+)$ to the ghost currents $j_+(j_-)$ given by:

$$T_{++} = \partial_+ c^a b_{+a}, \;\; T_{--} = \partial_- \tilde{c}^a b_{-a} \tag{3.0. 4}$$

$$j_+^i = b_{+a}(T^i)^a_b c^b, \;\; j_-^i = b_{-a}(T^i)^a_b \tilde{c}^b \tag{3.0. 5}$$

If one keeps track of the $i\epsilon$ prescriptions in the propagators appearing in the graphs, the loop integrals are finite, and we can write their contributions to the effective action as:

$$W^{(2)} = \frac{1}{4\pi} Tr T^a \int d^2p (h_{--}(p)\frac{p_+^2}{p_-}A_-^a(-p) + h_{++}(p)\frac{p_-^2}{p_+}A_+^a(-p)). \tag{3.0. 6}$$

The coefficient of (3.0. 6) and $W^{(2)}$ may also be computed using the OPE:

$$T(z)j_a(w) \sim \frac{Tr T_a}{(z-w)^3} + \frac{1}{(z-w)^2}j_a(w) + \frac{1}{(z-w)}\partial j_a(w). \tag{3.0. 7}$$

As it stands, $W^{(2)}$ has a gravitational anomaly, i.e. the energy-momentum tensor is not conserved. However we can still add local counterterms to (3.0. 6) to recover general coordinate invariance. Since to first order in h the two-dimensional scalar curvature has as Fourier transform:

$$R(p) = 2(2p_+p_-h_{+-}(p) - p_+^2 h_{--} - p_-^2 h_{++}), \tag{3.0. 8}$$

if we add the counterterms:

$$\begin{aligned} W_{\text{c.t.}} &= \frac{1}{4\pi} Tr T_a \int A_-^a(-p)(h_{++}(p)p_- - 2p_+h_{+-}) \\ &+ \frac{1}{4\pi} Tr T_a \int A_+^a(-p)(h_{--}(p)p_+ - 2p_-h_{+-}), \end{aligned} \tag{3.0. 9}$$

we obtain an effective action

$$W^{(2)} = \frac{1}{16\pi} Tr T_a \int R(p) \frac{p_+ A^a_-(-p) + p_- A^a_+(-p)}{p_+ p_-}, \qquad (3.0.\ 10)$$

leading to a conserved energy-momentum tensor, although it contains a trace anomaly which is not proportional to $R(p)$ and therefore it cannot be absorbed in a modification of the dilaton transformation. Varying (3.0. 10) with respect to h_{+-} leads to:

$$\langle T_{+-} \rangle = \frac{\delta W^{(2)}}{\delta h_{+-}} = \frac{1}{4\pi} Tr T_a (p_+ A^a_-(-p) + p_- A^a_+(-p)) \qquad (3.0.\ 11)$$

which in covariant form becomes $\sim Tr T_a \nabla^\alpha A^a_\alpha$.

Similarly we can vary the effective action to this order with respect to gauge transformations to evaluate the corresponding gauge anomaly:

$$\begin{aligned} &(D_- \frac{\delta W^{(2)}}{\delta A^a_-} + D_+ \frac{\delta W^{(2)}}{\delta A^a_+}) \sim p_- \frac{\delta W^{(2)}}{\delta A^a_-(p)} + p_+ \frac{\delta W^{(2)}}{\delta A^a_+(p)} \\ &= p_- \langle j_{a+}(p) \rangle + p_+ \langle j_{a-}(p) \rangle = -\frac{1}{8\pi} Tr T_a R(-p). \end{aligned} \qquad (3.0.\ 12)$$

This is a different way of writing the third order pole in the OPE (3.0. 7). From (3.0. 11) we see that at this order ($W^{(2)}$) the trace anomaly is not proportional to R, and it therefore cannot be absorbed in a contribution to the dilaton or the effective value of c (the central charge of the Virasoro algebra). The contribution in (3.0. 11) spoils the conformal invariance of the dual theory, and further fields should be required to cancel it. However in that case the resulting theory would not agree with the one obtained through a naive duality transformation. Another way to obtain the same conclusion as in (3.0. 11) is to use heat kernel methods. Both methods agree and we conclude that the condition for the duality transformation to respect conformal invariance is that the generators of the duality group in the adjoint representation should have a vanishing trace. The opposite may only happen for non-semisimple groups, as in the example discussed in [24].

4 The Transformation of the Dilaton

It is well known [16] that the transformation (2.1. 5) is not the whole story. Indeed, the dual model is not even conformally invariant in general, unless an appropriate transformation of the dilaton is included, namely

$$\tilde{\phi} = \phi - \frac{1}{2} \log k^2. \qquad (4.0.\ 1)$$

Perhaps the simplest way to realize that something has to change in the dilaton coupling is to insist on the demand that the BRS charge be nilpotent. It is well-known [33, 34] that the BRS charge can be written as:

$$Q = \oint \frac{dz}{2\pi i} c(z) (T^{(x)} + \frac{1}{2} T_{gh}), \qquad (4.0.\ 2)$$

where

$$T^{(x)} \equiv -\frac{1}{2} g_{\mu\nu} \partial x^\mu \partial x^\nu + \frac{1}{2}\partial^2 \phi$$
$$T_{gh} \equiv -2b\partial c - \partial b c \qquad (4.0.\ 3)$$

and $Q^2 = 0$ (in the OPE sense) is equivalent to the consistency conditions of the σ-model β-functions equal to zero [35].

Using the fact that after performing a duality transformation

$$\tilde{T}^{(x)} = T^{(x)} + \frac{1}{2k^2}[(k.\partial x)^2 - ((v-w).\partial x)^2] + \frac{1}{2}\partial^2(\tilde{\phi} - \phi) \qquad (4.0.\ 4)$$

the condition $\tilde{Q}^2 = 0$ necessarily leads to (4.0. 1).

We can trace the need for a transformation of the dilaton to the behavior of the measure under conformal transformations. Under a Weyl rescaling of the 2-d world-sheet metric, $g \to e^\sigma g$, the integration measure over the embeddings behaves (to first order in σ) as:

$$\mathcal{D}_{(e^\sigma g)} x = \mathcal{D}_g x \, e^{\frac{d}{48\pi} S_L(\sigma) + 6\alpha' \int (-\nabla^2\phi + (\nabla\phi)^2 - \frac{1}{4}R + \frac{1}{48}H^2)\sigma}, \qquad (4.0.\ 5)$$

where $S_L(\sigma)$ is the Liouville action. This means that although they are formally the same, both measures $\mathcal{D}x$ and $\mathcal{D}\tilde{x}$ behave in a very different way under Weyl transformations unless, of course, a compensating transformation of the dilaton is introduced to this purpose.

In the path integral approach the way to obtain the correct dilaton shift yielding to a conformally invariant dual theory can be seen as follows. Let us work with the approach of Roček and Verlinde. In complex coordinates and on spherical world sheets we can parametrize $A = \partial\alpha$, $\bar{A} = \bar{\partial}\beta$ (as we previously did in (3.0. 1)), for some 0-forms α, β in the manifold M. The change of variables from $A, \bar{A}$ to α, β produces a factor in the measure:

$$\mathcal{D}A\mathcal{D}\bar{A} = \mathcal{D}\alpha\mathcal{D}\beta(\det\partial)(\det\bar{\partial}) = \mathcal{D}\alpha\mathcal{D}\beta(\det\Delta). \qquad (4.0.\ 6)$$

Substituting $A, \bar{A}$ as functions of α, β in (2.1. 8) and integrating on α, β, the following determinant emerges:

$$(\det(\partial g_{00}\bar{\partial}))^{-1}. \qquad (4.0.\ 7)$$

In particular, the integration on β produces a delta-function

$$\delta(\bar{\partial}(g_{00}\partial\alpha + (g_{0\alpha} - b_{0\alpha})\partial x^\alpha - \partial\tilde{\theta})), \qquad (4.0.\ 8)$$

which integrated on α yields the factor in the measure (4.0. 7).

What we finally get in the measure is then

$$\frac{\det\Delta}{\det\Delta_{g_{00}}} \qquad (4.0.\ 9)$$

where $\Delta_{g_{00}}$ is given as in (2.1. 7) in complex notation. This formula provides a justification for Buscher's prescription (see also [30]) for the computation of the determinant arising from the naive gaussian integration. As we have just seen some care is needed in order to correctly define the measure of integration over the gauge fields. From (4.0. 9) the

dilaton shift (2.1. 6) is obtained in the following way. Writing g_{00} as $g_{00} = 1 + \sigma \approx e^{\sigma}$ we have:

$$\Delta_{g_{00}} = (1+\sigma)\Delta - h^{\mu\nu}\partial_\mu\sigma\partial_\nu. \qquad (4.0.\ 10)$$

Substituting in the infinitesimal variation of Schwinger's formula:

$$\delta \log \det\Delta = Tr \int_\epsilon^\infty dt \delta\Delta e^{-t\Delta} \qquad (4.0.\ 11)$$

we obtain

$$\delta \log \det\Delta_{g_{00}} = -\int d^2\xi\sqrt{h}\Omega\langle\xi|e^{-\epsilon(\Delta+\sigma\Delta-h^{\mu\nu}\partial_\mu\sigma\partial_\nu)}|\xi\rangle, \qquad (4.0.\ 12)$$

where $\delta\Delta_{g_{00}} = -\Omega\Delta_{g_{00}}$ with $\delta h_{\mu\nu} = \Omega\delta_{\mu\nu}$. We can now use the heat kernel expansion [31]:

$$\langle\xi|e^{-\epsilon D}|\xi\rangle = \frac{1}{4\pi\epsilon} + \frac{1}{4\pi}(\frac{1}{6}R^{(2)} - V), \qquad (4.0.\ 13)$$

where

$$D \equiv \Delta - 2ih^{\mu\mu}A_\mu\partial_\nu + (-\frac{i}{\sqrt{h}}\partial_\mu(\sqrt{h}h^{\mu\nu}A_\nu) + h^{\mu\nu}A_\mu A_\nu) + V. \qquad (4.0.\ 14)$$

For $D = \Delta + \sigma\Delta - h^{\mu\nu}\partial_\mu\sigma\partial_\nu$ and after dropping the divergent term $1/4\pi\epsilon$ and the quadratic terms in σ we obtain:

$$\delta \log \det\Delta_{g_{00}} = -\frac{1}{8\pi}\int d^2\xi\sqrt{h}R^{(2)}\log g_{00}. \qquad (4.0.\ 15)$$

Substituting in (2.1. 7):

$$\det g_{00} = \exp\,(-\frac{1}{8\pi}\int d^2\xi\sqrt{h}R^{(2)}\log g_{00}), \qquad (4.0.\ 16)$$

which implies $\tilde{\phi} = \phi - \frac{1}{2}\log g_{00}$.

5 Duality and the Cosmological Constant

A striking feature of duality is the fact that the cosmological constant, defined as the asymptotic value of the scalar curvature, is not in general invariant under the transformation. This fact was first noticed in [36] for the case of a WZW model with group $\widetilde{SL(2,R)}$ where a discrete subgroup was gauged. This space has negative cosmological constant and under a given duality transformation it is mapped into an asymptotically flat space (into a black string). This implies that the usual definition of the cosmological constant from the low-energy effective action is not satisfactory. Even at large distances, if duality is not broken there is a symmetry between local (momentum) modes and non-local (winding) modes. One is lead to wonder to what extent the cosmological constant is a string observable[3]. The contribution to the cosmological constant of the massless sector might be cancelled by the tower of massive states always present in String Theory (proposals along these lines using the Atkin-Lehner symmetry were advanced by G. Moore [39]).

[3]Similar remarks would apply to the concept of spacetime singularity in String Theory [37, 38].

We study now the behavior of the scalar curvature under duality. If the space-time metric in the σ-model takes the form

$$ds^2 = g_{ij}dx^i dx^j \qquad i,j = 0,1,2,...,d-1, \tag{5.0. 1}$$

where x^0 is adapted to the isometry $\vec{k} = \partial/\partial x^0$, (5.0. 1) can be written as

$$\begin{aligned} ds^2 &= (e^0)^2 + (g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2})dx^\alpha dx^\beta \\ e^0 &= k dx^0 + \frac{k_\alpha}{k} dx^\alpha \\ k^2 &= k_i k^i = g_{00} \qquad k_\alpha = g_{0\alpha}. \end{aligned} \tag{5.0. 2}$$

Buscher's transformation leads to a dual metric

$$\begin{aligned} d\tilde{s}^2 &= (\tilde{e}^0)^2 + (g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2})dx^\alpha dx^\beta \\ \tilde{e}^0 &= \frac{1}{k}(d\tilde{x}^0 + v_\alpha dx^\alpha), \end{aligned} \tag{5.0. 3}$$

$\tilde{x}^0$ being the Lagrange multiplier and v is defined as in section 2 by $k^l H_{lij} = -\partial_{[i}v_{j]}, H = db$. The dual scalar curvature following from (5.0. 3) is

$$\tilde{R} = R - \frac{4}{k^2}g^{\alpha\beta}\partial_\alpha k \partial_\beta k + \frac{4}{k}\Delta_q^{d-1}k + \frac{1}{k^2}H_{0\alpha\beta}H^{0\alpha\beta} - \frac{k^2}{4}F_{\alpha\beta}F^{\alpha\beta}, \tag{5.0. 4}$$

where Δ_q^{d-1} is the $(d-1)$-dimensional Laplacian for the metric $g^q_{\alpha\beta} = g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$, and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ with $A_\alpha = k_\alpha/k^2$. (5.0. 4) can be rewritten as

$$\tilde{R} = R + 4\Delta \log k + \frac{1}{k^2}H_{0\alpha\beta}H^{0\alpha\beta} - \frac{k^2}{4}F_{\alpha\beta}F^{\alpha\beta}. \tag{5.0. 5}$$

From (5.0. 5) we see that:

- The only way to "flatten" negative curvature is by having torsion in the initial space-time. Otherwise the dual of an asymptotically negatively curved space time is a space of the same type.
- Positive curvature seems easier to flatten.
- In general the asymptotic behaviors of $\tilde{R}$ and R are different, which proves the statement at the beginning of this section.
- In the particular case of constant toroidal compactifications $\tilde{R} = R$, in agreement with the result in [40].

We can also construct the dual torsion

$$\begin{aligned} \tilde{H}_{0\alpha\beta} &= -\frac{1}{2}F_{\alpha\beta} \\ \tilde{H}_{\alpha\beta\rho} &= H_{\alpha\beta\rho} - \frac{3}{k^2}H_{0[\alpha\beta}k_{\rho]} - \frac{3}{2}F_{[\alpha\beta}v_{\rho]}. \end{aligned} \tag{5.0. 6}$$

Since

$$\sqrt{g} = k^2 \sqrt{\tilde{g}}, \tag{5.0. 7}$$

and the modulus of k can be expressed in terms of the dilaton transformation properties,

$$\tilde{\phi} = \phi - \log k, \tag{5.0. 8}$$

we obtain

$$\tilde{R} + e^{2(\phi-\tilde{\phi})} \tilde{H}^2_{0\alpha\beta} + \Delta\tilde{\phi} = R + e^{2(\tilde{\phi}-\phi)} H^2_{0\alpha\beta} + \Delta\phi, \tag{5.0. 9}$$

which could be used to show the duality invariance of the string effective action to leading order in α'.

The change of the cosmological constant under duality is not only peculiar to three-dimensions [36] but rather generic. This raises the physical question of whether in the context of String Theory the value of the cosmological constant can be inferred from the asymptotic (long distance) behavior of the Ricci tensor. If duality is not broken, the answer seems to be in the negative, and it makes the issue of what is the correct meaning of the cosmological constant in String Theory yet more misterious.

6 The Physical Definition of Distance

The existence of duality raises the question of the empirical definition of distance. This is, of course, not a well defined question in the absence of a sufficiently developed String Field Theory, but can nevertheless be asked if we assume that the outcome of every possible experiment is some correlation function of the corresponding two dimensional CFT.

Thinking on the simplest situation of closed strings propagating in a spacetime with one coordinate compactified in a circle, it is physically obvious that if we attempt to measure distances through the asymptotic behavior of correlation functions at large separations[4] [41], we would get a completely different answer if we use pure momentum states (of energy $E_p = n/R$) or pure windings states (of energy $E_w = mR$), which we would most simply reinterpret as momentum states of a torus of radius $1/R$. This would lead us in a natural way to restrict the allowed outcome of our experiments to the interval $d \in (1, \infty)$.

There are some technical complications, stemming from the fact that the Polyakov method only allows to compute on-shell correlators, which means that we cannot probe directly off-shell amplitudes.

In the absence of any clear physical distinction among different classes of states, perhaps the most natural possibility is to define distances out of "unpolarized" correlators, that is, considering the contribution of all states at the same time.

There is still a certain freedom as to how to perform the corresponding Fourier transform in order to define physical quantities in position space. The most sensible thing seems, however, to make use of the fact that momentum and winding states define a lattice [11, 12]. To be specific, sticking for concreteness to the case in which r dimensions

[4] At large spatial distances the propagator behaves as $G(\vec{r}, \vec{r}^{\,'}; M) \sim e^{-M|\vec{r}-\vec{r}^{\,'}|}$ so that a suitable definition of distance is given by $d(\vec{r}, \vec{r}^{\,'}) \equiv \frac{1}{M} \log \frac{G(\vec{r},\vec{r}^{\,'};M)}{G(\vec{r},\vec{r}^{\,'};2M)}$.

(called $\vec{x}$) are compactified in circles of radius R, and denoting by $\vec{y}$ the $(d-r)$-dimensional set of all other coordinates, the above considerations yield:

$$G(\vec{x},\vec{x}^{'};\vec{y}-\vec{y}^{'},t-t^{'}) \equiv \sum_{\vec{n},\vec{m}} e^{2\pi i(\vec{x}-\vec{x}^{'})(\vec{n}/R+\vec{m}R/2)} \int dp_0 d\mu(\vec{p}) e^{i\vec{p}(\vec{y}-\vec{y}^{'})-ip_0(t-t^{'})} \langle V_{\vec{n},\vec{m}} V_{-\vec{n},-\vec{m}} \rangle f_{\vec{n},\vec{m}}, \tag{6.0. 1}$$

where $V_{\vec{n},\vec{m}}$ represents the vertex operator corresponding to the sector with momentum numbers $\vec{n}$ and winding numbers $\vec{m}$, and we will moreover consider pure solitonic states, without any oscillators $N = \tilde{N} = 0$[5].

The momentum space correlator is then given essentially by the delta function implementing the condition that the vertex operator has conformal dimension 1, that is:

$$p_0^2 - \vec{p}^2 - (\frac{\vec{n}}{R} + \vec{m}\frac{R}{2})^2 = N - 1 + \tilde{N} - 1 = -2. \tag{6.0. 2}$$

A further restriction ($\vec{n}\vec{m} = 0$) comes from invariance under translations in σ ($L_0 = \tilde{L}_0$).

Using the integral representation for the delta functions, the integral over p_0 can be easily performed, and the double sum packed into a Riemann theta-function:

$$G(\vec{x},\vec{x}^{'};\vec{y}-\vec{y}^{'},t-t^{'}) = \int d\mu(\vec{p}) \int_{-\infty}^{\infty} d\tau \tau^{-1/2} \int_{-\infty}^{\infty} d\lambda e^{-i\frac{(t-t^{'})^2}{4\tau} - i\tau(\vec{p}^2-2) + i\vec{p}(\vec{y}-\vec{y}^{'})} \theta(\vec{z},\Omega), \tag{6.0. 3}$$

where $\vec{z} \equiv (\vec{x}-\vec{x}^{'})(\frac{R}{2},\frac{1}{R})$ and

$$\Omega \equiv \frac{1}{\pi} \begin{pmatrix} -\tau R^2/2 & \frac{(\lambda-\tau)}{2} \\ \frac{(\lambda-\tau)}{2} & -\frac{\tau}{R^2} \end{pmatrix}. \tag{6.0. 4}$$

A different expression can be obtained in terms of a double sum of $(d-r)$-dimensional Pauli-Jordan functions

$$G^{(d)}(\vec{x},\vec{x}^{'};\vec{y}-\vec{y}^{'},t-t^{'}) = \sum_{\vec{m},\vec{n}} G_{PJ}^{(d-r)}(\vec{y}-\vec{y}^{'},t-t^{'};M^2(\vec{n},\vec{m})) e^{2\pi i(\frac{\vec{n}}{R}+\vec{m}\frac{R}{2})(\vec{x}-\vec{x}^{'})} \delta(\vec{n}\vec{m}), \tag{6.0. 5}$$

where the "mass spectrum" is given by

$$M^2(\vec{n},\vec{m}) \equiv (\frac{\vec{n}}{R} + \vec{m}\frac{R}{2})^2 - 2. \tag{6.0. 6}$$

It is plain that any definition of distance based on the preceding ideas lacks any periodicity (which shows only in the particular cases in which pure winding states $f_{\vec{n},\vec{m}} = \delta_{\vec{n},\vec{0}}$ or pure momentum states $f_{\vec{n},\vec{m}} = \delta_{\vec{m},\vec{0}}$ are used).

It is also arguable whether these correlators are indeed the most natural ones to consider from the physical point of view. At extreme (either very high or very low) values of the radius, "pure" states (winding or momentum) are much lighter than all the others, so that it is perhaps more natural to define distances in terms of the lightest states only [42].

One could always consider our suggestion as a concrete implementation of earlier speculations that at very short distances there could be a physical regime at which geometry ceases to be smooth, but distances can nevertheless be defined, and they obey the triangular inequality [43].

[5] In the unpolarized case we are favouring, the selection function is trivial $f_{\vec{n},\vec{m}} = 1$, but we have written it in the formula in order to allow for more general possibilities.

7 The Canonical Approach

The procedures to implement duality explained in section 2 look unnecessarily complicated. In the one due to Roček and Verlinde the isometry is gauged, the (non propagating) gauge fields are constrained to be trivial, and the Lagrange multipliers themselves are promoted to the rank of new coordinates once the gaussian integration over the gauge fields is performed. One suspects that all those complicated intermediate steps could be avoided, and that it should be possible to pass directly from the original to the dual theory.

Some suggestions have indeed been made in the literature pointing (at least in the simplified situation where all backgrounds are constant or dependent only on time) towards an understanding of duality as particular instances of canonical transformations [14, 44].

In this section we are going to show that this idea works well when the background admits an abelian isometry [45], laying duality on a simpler setting than before, namely as a (privileged) subgroup of the whole group of (non-anomalous, that is implementable in Quantum Field Theory [46]) canonical transformations on the phase space of the theory.

We will proof that Buscher's transformation formulae can be derived by performing a given canonical transformation on the Hamiltonian of the initial theory. We believe that this is a "minimal" approach in the sense that no extraneous structure has to be introduced, and all standard results in the abelian case (and more) are easily recovered using it. In particular it is possible to perform the duality transformation in arbitrary coordinates not only in the original manifold (which was also possible in Roček and Verlinde's formulation) but also in the dual one. The multivaluedness and periods of the dual variables can be easily worked out from the implementation of the canonical transformation in the path integral. The generalization to arbitrary genus Riemann surfaces is in this approach straightforward. The behavior of currents not commuting with those used to implement duality can also be clarified. In the case of WZW models it becomes rather simple to prove that the full duality group is given by $Aut(G)_L \times Aut(G)_R$, where L, R refer to the left- and right-currents on the model with group G, and $Aut(G)$ are the automorphisms of G, both inner and outer. Due to the chiral conservation of the currents in this case, the canonical transformation leads to a local expression for the dual currents. In the case where the currents are not chirally conserved, then those currents associated to symmetries not commuting with the one used to perform duality become generically non-local in the dual theory and this is why they are not manifest in the dual Lagrangian. All the generators of the full duality group $O(d, d; Z)$ can be described in terms of canonical transformations. This gives the impression that the duality group should be understood in terms of global symplectic diffemorphisms. It would be useful to formulate it in the context of some analogue of the group of disconnected diffeomorphisms, but for the time being such a construction is lacking.

Concerning non-abelian duality, it seems to fall beyond the scope of the Hamiltonian point of view. There is one example [47] in which the non-abelian dual has been constructed out of a canonical transformation but it is still early to say whether the general case can be treated similarly.

7.1 The Abelian Case

We start with a bosonic sigma model written in arbitrary coordinates on a manifold M with Lagrangian

$$L = \frac{1}{2}(g_{ab} + b_{ab})(\phi)\partial_+\phi^a\partial_-\phi^b \tag{7.1. 1}$$

where $x^{\pm} = (\tau \pm \sigma)/2$, $a, b = 1, \ldots, d = \dim M$. The corresponding Hamiltonian is

$$H = \frac{1}{2}(g^{ab}(p_a - b_{ac}\phi^{'\,c})(p_b - b_{bd}\phi^{'\,d}) + g_{ab}\phi^{'\,a}\phi^{'\,b}) \tag{7.1. 2}$$

where $\phi^{'\,a} \equiv d\phi^a/d\sigma$. We assume moreover that there is a Killing vector field k^a, $\mathcal{L}_k g_{ab} = 0$ and $i_k H = -dv$ for some one-form v, where $(i_k H)_{ab} \equiv k^c H_{cab}$ and $H = db$ locally. This guarantees the existence of a particular system of coordinates, "adapted coordinates", which we denote by $x^i \equiv (\theta, x^\alpha)$, such that $\vec{k} = \partial/\partial\theta$. We denote the jacobian matrix by $e^i_a \equiv \partial x^i/\partial\phi^a$.

This defines a point transformation in the original Lagrangian (7.1. 1) which acts on the Hamiltonian as a canonical transformation with generating function $\Phi = x^i(\phi)p_i$, and yields:

$$\begin{aligned} p_a &= e^i_a p_i \\ x^i &= x^i(\phi). \end{aligned} \tag{7.1. 3}$$

Once in adapted coordinates we can write the sigma model Lagrangian as

$$L = \frac{1}{2}G(\dot\theta^2 - \theta^{'\,2}) + (\dot\theta + \theta^{'})J_- + (\dot\theta - \theta^{'})J_+ + V \tag{7.1. 4}$$

where

$$\begin{aligned} G = g_{00} = k^2 \qquad V = \frac{1}{2}(g_{\alpha\beta} + b_{\alpha\beta})\partial_+x^\alpha\partial_-x^\beta \\ J_- = \frac{1}{2}(g_{0\alpha} + b_{0\alpha})\partial_-x^\alpha \qquad J_+ = \frac{1}{2}(g_{0\alpha} - b_{0\alpha})\partial_+x^\alpha. \end{aligned} \tag{7.1. 5}$$

In finding the dual with a canonical transformation we can use the Routh function with respect to θ, i.e. we only apply the Legendre transformation to $(\theta, \dot\theta)$. The canonical momentum is given by

$$p_\theta = G\dot\theta + (J_+ + J_-) \tag{7.1. 6}$$

and the Hamiltonian

$$\begin{aligned} H = p_\theta\dot\theta - L = \frac{1}{2}G^{-1}p_\theta^2 - G^{-1}(J_+ + J_-)p_\theta + \frac{1}{2}G\theta^{'\,2} + \\ + \frac{1}{2}G^{-1}(J_+ + J_-)^2 + \theta^{'}(J_+ - J_-) - V. \end{aligned} \tag{7.1. 7}$$

The Hamilton equations are:

$$\begin{aligned} \dot\theta &= \frac{\delta H}{\delta p_\theta} = G^{-1}(p_\theta - J_+ - J_-) \\ \dot{p_\theta} &= -\frac{\delta H}{\delta\theta} = (G\theta^{'} + J_+ - J_-)^{'} \end{aligned} \tag{7.1. 8}$$

and the current components:

$$\mathcal{J}_+ = \frac{1}{2}G\partial_+\theta + J_+ = \frac{1}{2}p_\theta + \frac{1}{2}G\theta' + \frac{J_+ - J_-}{2}$$
$$\mathcal{J}_- = \frac{1}{2}G\partial_-\theta + J_- = \frac{1}{2}p_\theta - \frac{1}{2}G\theta' - \frac{J_+ - J_-}{2}. \qquad (7.1.\ 9)$$

It can easily be seing that the current conservation $\partial_-\mathcal{J}_+ + \partial_+\mathcal{J}_- = 0$ is equivalent to the second Hamilton equation $\dot{p_\theta} = -\delta H/\delta\theta$.

The generator of the canonical transformation we choose is:

$$F = \frac{1}{2}\int_{D,\partial D=S^1} d\tilde{\theta}\wedge d\theta = \frac{1}{2}\oint_{S^1}(\theta'\tilde{\theta} - \theta\tilde{\theta}')d\sigma \qquad (7.1.\ 10)$$

that is,

$$p_\theta = \frac{\delta F}{\delta\theta} = -\tilde{\theta}'$$
$$p_{\tilde{\theta}} = -\frac{\delta F}{\delta\tilde{\theta}} = -\theta'. \qquad (7.1.\ 11)$$

This generating functional does not receive any quantum corrections (as explained in [46]) since it is linear in θ and $\tilde{\theta}$. If θ was not an adapted coordinate to a continuous isometry, the canonical transformation would generically lead to a non-local form of the dual Hamiltonian. Since the Lagrangian and Hamiltonian in our case only depend on the time- and space-derivatives of θ, there are no problems with non-locality. The transformation (7.1. 11) in (7.1. 7) gives:

$$\tilde{H} = \frac{1}{2}G^{-1}\tilde{\theta}'^{\,2} + G^{-1}(J_+ + J_-)\tilde{\theta}' +$$
$$\frac{1}{2}Gp_{\tilde{\theta}}^2 - (J_+ - J_-)p_{\tilde{\theta}} + \frac{1}{2}G^{-1}(J_+ + J_-)^2 - V. \qquad (7.1.\ 12)$$

Since:

$$\dot{\tilde{\theta}} = \frac{\delta\tilde{H}}{\delta p_{\tilde{\theta}}} = Gp_{\tilde{\theta}} - (J_+ - J_-), \qquad (7.1.\ 13)$$

we can perform the inverse Legendre transform:

$$\tilde{L} = \frac{1}{2}G^{-1}(\dot{\tilde{\theta}}^2 - \tilde{\theta}'^{\,2}) + G^{-1}J_+(\dot{\tilde{\theta}} - \tilde{\theta}')$$
$$-G^{-1}J_-(\dot{\tilde{\theta}} + \tilde{\theta}') + V - 2G^{-1}J_+J_-. \qquad (7.1.\ 14)$$

From this expression we can read the dual metric and torsion and check that they are given by Buscher's formulae[6]:

$$\tilde{g}_{00} = 1/g_{00}, \qquad \tilde{g}_{0\alpha} = -b_{0\alpha}/g_{00}, \qquad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta}}{g_{00}}$$
$$\tilde{b}_{0\alpha} = -\frac{g_{0\alpha}}{g_{00}}, \qquad \tilde{b}_{\alpha\beta} = b_{\alpha\beta} - \frac{g_{0\alpha}b_{0\beta} - g_{0\beta}b_{0\alpha}}{g_{00}} \qquad (7.1.\ 15)$$

[6]The minus signs in $\tilde{g}_{0\alpha}$ and $\tilde{b}_{0\alpha}$ can be absorbed in a redefinition $\tilde{\theta}\to -\tilde{\theta}$.

For the dual theory to be conformal invariant the dilaton must transform as $\Phi' = \Phi - \frac{1}{2}\log g_{00}$ [16] [10]. We have not been able to find any argument justifying this transformation within the canonical transformations approach.

The dual manifold $\tilde{M}$ is automatically expressed in coordinates adapted to the dual Killing vector $\tilde{\vec{k}} = \partial/\partial\tilde{\theta}$. We can now perform another point transformation, with the same jacobian as (7.1. 3) to express the dual manifold in coordinates which are as close as possible to the original ones.

The transformations we perform are then: First a point transformation $\phi^a \to \{\theta, x^\alpha\}$, to go to adapted coordinates in the original manifold. Then a canonical transformation $\{\theta, x^\alpha\} \to \{\tilde{\theta}, x^\alpha\}$, which is the true duality transformation. And finally another point transformation $\{\tilde{\theta}, x^\alpha\} \to \tilde{\phi}^a$, with the same jacobian as the first point transformation, to express the dual manifold in general coordinates.

It turns out that the composition of these three transformations can be expressed in geometrical terms using only the Killing vector k^a, $\omega_a \equiv e^0_a$ and the corresponding dual quantities[7].

It is then quite easy to check that the total canonical transformation to be made in (7.1. 1) is just

$$\begin{aligned} k^a p_a &\to \tilde{\omega}_a \tilde{\phi}'^{\,a} \\ \omega_a \phi'^{\,a} &\to \tilde{k}^a \tilde{p}_a, \end{aligned} \tag{7.1. 16}$$

whose generating function is[8]

$$F = \frac{1}{2}\int_D \tilde{\omega}\wedge\omega = \frac{1}{2}\int_D \tilde{\omega}_a d\tilde{\phi}^a \wedge \omega_b d\phi^b. \tag{7.1. 17}$$

One then easily performs the transformations in such a way that the dual metric and torsion can be expressed in geometrical terms as

$$\tilde{g}_{ab} = g_{ab} - \frac{1}{k^2}(k_a k_b - (v_a - \omega_a)(v_b - \omega_b)) \tag{7.1. 18}$$

$$\tilde{g}^{ab} = g^{ab} + \frac{1}{(1+k.v)^2}[(k^2 + (v-\omega)^2)k^a k^b - 2(1+k.v)(k^{(a}(v-\omega)^{b)}] \tag{7.1. 19}$$

and

$$\tilde{b}_{ab} = b_{ab} - \frac{2}{k^2}k_{[a}(v-\omega)_{b]}, \tag{7.1. 20}$$

where

$$\begin{aligned} k_{(a}(v-\omega)_{b)} &= \frac{1}{2}(k_a(v_b - \omega_b) + k_b(v_a - \omega_a)) \\ k_{[a}(v-\omega)_{b]} &= \frac{1}{2}(k_a(v_b - \omega_b) - k_b(v_a - \omega_a)). \end{aligned} \tag{7.1. 21}$$

[7] Note that we must raise and lower indices with the dual metric, i.e. $\tilde{e}_{ia} = \tilde{g}_{ij}\tilde{e}^j_a, \tilde{e}^{ia} = \tilde{g}^{ab}\tilde{e}^i_b$, which implies $\tilde{\omega}_a = \omega_a$, but $\tilde{\omega}^a = k^a(k^2+v^2) + \vec{e}^{\,a}\cdot v$ (where $\vec{e}^{\,a} \equiv e^a_\alpha$), $\tilde{k}^a = k^a$ but $\tilde{k}_a = (\omega_a - (\vec{e}_a \cdot v))/k^2$. We have moreover $\tilde{\omega}^{\,2} = k^2 + v^2 + g^{\alpha\beta}v_\beta\omega_\alpha$ and $\tilde{k}^2 = 1/k^2$.

[8] The one-form $\omega \equiv \omega_a d\phi^a$ is dual to the Killing vector $\vec{k}$: $\omega(\vec{k}) = 1$, $\omega(\vec{e}_\alpha) = 0$, but it is of course different from $\underline{k} \equiv k_a/k^2\, d\phi^a$ (the former is an exact form, whereas the latter does not even in general satisfy Frobenius condition $\underline{k}\wedge d\underline{k} = 0$).

These formulae are the covariant generalization of (7.1. 15). The canonical approach has been very useful in order to obtain the dual manifold in an arbitrary coordinate system. With the usual approaches it is expressed in adapted coordinates to the dual isometry. This happens because the dual variables appear as Lagrange multipliers and after an integration by parts only the derivatives of them emerge, being then adapted coordinates automatically.

Some other useful information can be extracted easier in the approach of the canonical transformation.

From the generating functional (7.1. 10) we can learn about the multivaluedness and periods of the dual variables [19]. Since θ is periodic and in the path integral the canonical transformation is implemented by [46]:

$$\psi_k[\tilde{\theta}(\sigma)] = N(k) \int \mathcal{D}\theta(\sigma) e^{iF[\tilde{\theta},\theta(\sigma)]} \phi_k[\theta(\sigma)] \tag{7.1. 22}$$

where $N(k)$ is a normalization factor, $\phi_k(\theta + a) = \phi_k(\theta)$ implies for $\tilde{\theta}$: $\tilde{\theta}(\sigma + 2\pi) - \tilde{\theta}(\sigma) = 4\pi/a$, which means that $\tilde{\theta}$ must live in the dual lattice of θ. Note that (7.1. 22) suffices to construct the dual Hamiltonian. It is a simple exercise to check that acting with (7.1. 12) on the left-hand side of (7.1. 22) and pushing the dual Hamiltonian through the integral we obtain the original Hamiltonian acting on $\phi_k[\theta(\sigma)]$:

$$\tilde{H}\psi_k[\tilde{\theta}(\sigma)] = N(k) \int \mathcal{D}\theta(\sigma) e^{iF[\tilde{\theta},\theta(\sigma)]} H\phi_k[\theta(\sigma)] \tag{7.1. 23}$$

This makes the duality transformation very simple conceptually, and it also implies how it can be applied to arbitrary genus Riemann surfaces, because the state $\phi_k[\theta(\sigma)]$ could be the state obtained by integrating the original theory on an arbitrary Riemann surface with boundary. It is also clear that the arguments generalize straightforwardly when we have several commuting isometries.

One can easily see that under the canonical transformation the Hamilton equations are interchanged:

$$\begin{aligned} \dot{p_\theta} &= -\frac{\delta H}{\delta \theta} = (G\theta' + J_+ - J_-)' \rightarrow \dot{\tilde{\theta}} = G p_{\tilde{\theta}} - J_+ + J_- \\ \dot{\theta} &= \frac{\delta H}{\delta p_\theta} = G^{-1}(p_\theta - J_+ - J_-) \rightarrow \dot{p_{\tilde{\theta}}} = (G^{-1}(\tilde{\theta}' + J_+ + J_-))', \end{aligned} \tag{7.1. 24}$$

and that the canonical transformed currents conservation law is in this case equivalent to the first Hamilton equation.

In the chiral case $J_- = 0$ (i.e. $g_{0i} = -b_{0i}$) and G is a constant, therefore we can normalize θ to set $G = 1$ and :

$$L = \frac{1}{2}(\dot{\theta}^2 - \theta'^{\,2}) + (\dot{\theta} - \theta')J_+ + V. \tag{7.1. 25}$$

The Hamiltonian is

$$H = \frac{1}{2}p_\theta^2 - J_+ p_\theta + \frac{1}{2}(J_+ + \theta')^2 - V. \tag{7.1. 26}$$

The action is invariant under $\delta\theta = \alpha(x^+)$, a $U(1)_L$ Kac-Moody symmetry. The $U(1)$ Kac-Moody algebra has the automorphism $\mathcal{J}_+ \to -\mathcal{J}_+$. This is precisely the effect of the canonical transformation. The equation of motion or current conservation is:

$$\partial_-(\partial_+\theta + J_+) = 0. \tag{7.1. 27}$$

$\mathcal{J}_+ = \partial_+\theta + J_+ = p_\theta + \theta'$ transforms under the canonical transformation in $\mathcal{J}_+^{c.t.} = -\tilde{\theta}' - p_{\tilde{\theta}} = -\mathcal{J}_+$.

One can also follow the transformation to the dual model of other continuous symmetries. The simplest case is as usual the WZW-model which is the basic model with chiral currents. Consider for simplicity the level-k $SU(2)$-WZW model with action

$$S[g] = \frac{-k}{2\pi}\int d^2\sigma Tr(g^{-1}\partial_+ g g^{-1}\partial_- g) + \frac{k}{12\pi}\int Tr(g^{-1}dg)^3. \tag{7.1. 28}$$

The left- and right-chiral currents are

$$\mathcal{J}_+ = \frac{k}{2\pi}\partial_+ g g^{-1} \qquad \mathcal{J}_- = -\frac{k}{2\pi}g^{-1}\partial_- g. \tag{7.1. 29}$$

Parametrizing g in terms of Euler angles

$$g = e^{i\alpha\sigma_3/2}e^{i\beta\sigma_2/2}e^{i\gamma\sigma_3/2}, \tag{7.1. 30}$$

$\mathcal{J}_+$ are given by:

$$\begin{aligned}
\mathcal{J}_+^1 &= \frac{k}{2\pi}(-\cos\alpha\sin\beta\partial_+\gamma + \sin\alpha\partial_+\beta) \\
\mathcal{J}_+^2 &= \frac{k}{2\pi}(\sin\alpha\sin\beta\partial_+\gamma + \cos\alpha\partial_+\beta) \\
\mathcal{J}_+^3 &= \frac{k}{2\pi}(\partial_+\alpha + \cos\beta\partial_+\gamma),
\end{aligned} \tag{7.1. 31}$$

and similarly for the right currents. If we perform duality with respect to $\alpha \to \alpha +$ constant, $\mathcal{J}_+^3 \to -\mathcal{J}_+^3, \mathcal{J}_-^3 \to \mathcal{J}_-^3$ since $\mathcal{J}_+^3$ is the current component adapted to the isometry. For these currents it is easy to find the action of the canonical transformation because only the derivatives of α appear. For $\mathcal{J}_+^{1,2}$ there is an explicit dependence on α and it seems that the transform of these currents is very non-local. However due to its chiral nature, one can show that there are similar chirally conserved currents in the dual model. To do this we first combine the currents in terms of root generators:

$$\begin{aligned}
\mathcal{J}_+^{(+)} &= \mathcal{J}_+^1 + i\mathcal{J}_+^2 = e^{-i\alpha}(i\partial_+\beta - \sin\beta\partial_+\gamma) = e^{-i\alpha}j_+^{(+)} \\
\mathcal{J}_+^{(-)} &= \mathcal{J}_+^1 - i\mathcal{J}_+^2 = -e^{i\alpha}(i\partial_+\beta + \sin\beta\partial_+\gamma) = e^{i\alpha}j_+^{(-)}.
\end{aligned} \tag{7.1. 32}$$

From chiral current conservation $\partial_-\mathcal{J}_+^{(\pm)} = 0$ we obtain

$$\partial_- j_+^{(\pm)} = \pm i\partial_-\alpha j_+^{(\pm)}. \tag{7.1. 33}$$

In these equations only $\dot{\alpha}, \alpha'$ appear, and after the canonical transformation we can reconstruct the dual non-abelian currents (in the previous equations the canonical transformation amounts to the replacement $\alpha \to \tilde{\alpha}$) which take the same form as the original

ones except that with respect to the transformed $\mathcal{J}_+^3$ the rôles of positive and negative roots get exchanged. One also verifies that $\mathcal{J}_-^a$ are unaffected. This implies therefore that the effect of duality with respect to shifts of α is an automorphism of the current algebra amounting to performing a Weyl transformation on the left currents only while the right ones remain unmodified. This result although known [21] is much easier to derive in the Hamiltonian formalism than in the Lagrangian formalism where one must introduce external sources which carry some ambiguities. The construction for $SU(2)$ can be straightforwardly extended to other groups. This implies that for WZW-models the full duality group is $Aut(G)_L \times Aut(G)_R$, where $Aut(G)$ is the group of automorphisms of the group G, including Weyl transformations and outer automorphisms. For instance if we take $SU(N)$, the transformation $J_+ \to -J_+^T$, i.e. charge conjugation, follows from a canonical transformation of the type discussed. It suffices to take as generating functions for the canonical transformation the sum of the generating functions for each generator in the Cartan subalgebra. It is important to remark that the chiral conservation of the currents is crucial to guarantee the locality of the dual non-abelian currents. If the conserved current with respect to which we dualize is not chirally conserved locality is not obtained. The simplest example to verify this is the principal chiral model for $SU(2)$, which although is not a CFT serves for illustrative purposes. The equations of motion for this model imply the conservation laws:

$$\partial_- \mathcal{J}_+^a + \partial_+ \mathcal{J}_-^a = 0 \tag{7.1. 34}$$

where

$$\mathcal{J}_\pm = \frac{k}{2\pi} \partial_\pm g g^{-1}. \tag{7.1. 35}$$

If we perform duality with respect to the invariance under α translations we know how $\mathcal{J}_\pm^3$ transform, since they are the currents associated to the isometry. With the canonical transformation is possible to see as well which are the other dual conserved currents. Since the dual model is only $U(1)$-invariant one expects the rest of the currents to become non-local [47]. In terms of the root generators introduced in (7.1. 32) the conservation laws

$$\partial_- \mathcal{J}_+^{(\pm)} + \partial_+ \mathcal{J}_-^{(\pm)} = 0 \tag{7.1. 36}$$

are expressed:

$$\partial_- j_+^{(\pm)} + \partial_+ j_-^{(\pm)} \mp i(\partial_- \alpha j_+^{(\pm)} + \partial_+ \alpha j_-^{(\pm)}) = 0. \tag{7.1. 37}$$

Performing the canonical transformation we obtain that the dual conserved currents are given by:

$$\begin{aligned} \tilde{\mathcal{J}}_\pm^{(+)} &= \exp\left(i \int d\sigma (\dot{\tilde{\alpha}} + \cos\beta\gamma')\right)(i\partial_\pm \beta - \sin\beta \partial_\pm \gamma) \\ \tilde{\mathcal{J}}_\pm^{(-)} &= -\exp\left(-i \int d\sigma (\dot{\tilde{\alpha}} + \cos\beta\gamma')\right)(i\partial_\pm \beta + \sin\beta \partial_\pm \gamma) \end{aligned} \tag{7.1. 38}$$

which cannot be expressed in a local form.

7.2 The Non-Abelian Case

In view of the simplicity of the canonical approach to abelian duality, one could be tempted to think that the corresponding generalization to the non-abelian case would not be very

difficult. Unfortunately this is not the case, the reason being that there are no adapted coordinates to a set of non-commuting isometries, and therefore one is led to a non-local form of the Hamiltonian. In [25] we could carry out the non-abelian duality transformation due to the existence of chiral currents and as a consequence of the Polyakov-Wiegmann property [50] satisfied by WZW-actions. Although in the intermediate steps it was necessary to introduce non-local variables, the final result led to a local action in the new variables as a result of the special properties of WZW-models mentioned. The computations could be carried out exactly until the end to evaluate the form of the effective action in terms of the auxiliary variables needed in the construction of non-abelian duals. We have so far been unable to express these functional integral manipulations in a Hamiltonian setting as in the previous section.

To finish this section we present an example from the literature in which a canonical transformation produces a given non-abelian dual model. This example was presented in [47]. They consider the principal chiral model with group $SU(2)$ and construct a local canonical transformation mapping the model in a theory which turns out to be the non-abelian dual with respect to the left action of the whole group. This example was studied in the context of non-abelian duality in [48, 19].

The initial theory is the principal chiral model defined by the Lagrangian:

$$L = Tr(\partial_\mu g \partial^\mu g^{-1}), \tag{7.2. 1}$$

where $g \in SU(2)$. Parametrizing $g = \phi^0 + i\sigma^j\phi^j$, with ϕ^0, ϕ^j subject to the constraint $(\phi^0)^2 + \phi^2 = 1$ and $\phi^2 \equiv \sum_j (\phi^j)^2$, (7.2. 1) becomes:

$$L = \frac{1}{2}(\delta^{ij} + \frac{\phi^i\phi^j}{1-\phi^2})\partial_\mu\phi^i\partial^\mu\phi^j. \tag{7.2. 2}$$

The generating functional:

$$F[\psi,\phi] = \int_{-\infty}^{+\infty} dx\psi^i(\sqrt{1-\phi^2}\frac{\partial}{\partial x}\phi^i - \phi^i\frac{\partial}{\partial x}\sqrt{1-\phi^2} + \epsilon^{ijk}\phi^j\frac{\partial}{\partial x}\phi^k) \tag{7.2. 3}$$

produces the canonical transformation:

$$\begin{aligned} p_i &= \frac{\delta F[\psi,\phi]}{\delta\psi^i} = (\sqrt{1-\phi^2}\delta^{ij} + \frac{\phi^i\phi^j}{\sqrt{1-\phi^2}} - \epsilon^{ijk}\phi^k)\frac{\partial}{\partial x}\phi^j \\ \tilde{p}_i &= -\frac{\delta F[\psi,\phi]}{\delta\phi^i} = (\sqrt{1-\phi^2}\delta^{ij} + \frac{\phi^i\phi^j}{\sqrt{1-\phi^2}} + \epsilon^{ijk}\phi^k)\frac{\partial}{\partial x}\psi^j + \\ &\quad (\frac{2}{\sqrt{1-\phi^2}}(\phi^i\psi^j - \psi^i\phi^j) - 2\epsilon^{ijk}\psi^k)\frac{\partial}{\partial x}\phi^j, \end{aligned} \tag{7.2. 4}$$

which transforms (7.2. 2) into:

$$\tilde{L} = \frac{1}{1+4\psi^2}[\frac{1}{2}(\delta^{ij} + 4\psi^i\psi^j)\partial_\mu\psi^i\partial^\mu\psi^j - \epsilon^{\mu\nu}\epsilon^{ijk}\psi^i\partial_\mu\psi^j\partial_\nu\psi^k]. \tag{7.2. 5}$$

This is the non-abelian dual of (7.2. 1) with respect to the left action of the whole group [48, 19]. The generating functional (7.2. 3) can be written as:

$$F[\psi,\phi] = \int_{-\infty}^{+\infty} dx\psi^i J_i^1[\phi] \tag{7.2. 6}$$

where $J_i^1[\phi]$ are the spatial components of the conserved currents of the initial theory. (7.2. 6) is linear in the dual variables but not in the initial ones, so it will receive quantum corrections when implemented in the path integral. From (7.2. 4) it is not obvious that the dual model will not depend on the original variables ϕ^i. However this is so. Whether this way of constructing the generating functional of non-abelian duality is general or only works in this particular example is still an open question.

8 Conclusions and Open Problems

In these lectures a general exposition of abelian and non-abelian duality has been given. The usual approaches in the literature to both kinds of dualities have been reviewed. The goals of these approaches have been also exhibited, and some of them derived explicitly, as the formulation of abelian duality in an arbitrary coordinate system. The canonical transformations approach to abelian duality presented in [45] has been studied in detail, focussing especially in the problems that could not be solved in the usual approaches of Buscher or Roček and Verlinde or were difficult to study. The non-abelian case has been also considered, although the general construction as a canonical transformation is not yet understood. As was mentioned in the lectures the example given by Curtright and Zachos in [47] opens the possibility for non-abelian duality to be formulated in this way, in spite of the difficulties already mentioned concerning the impossibility of finding an adapted coordinate system to the whole set on non-commuting isometries.

The relation between duality and external automorphisms [51, 52, 53] is also much in need of further clarification.

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Some aspects of the cosmological conformal equivalence between "Jordan Frame" and "Einstein Frame"

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Abstract

The conformal equivalence between Jordan frame and Einstein frame can be used in order to search for exact solutions in general theories of gravity in which scalar fields are minimally or nonminimally coupled with geometry. In the cosmological arena a relevant role is played by the time parameter in which dynamics is described. In this paper we discuss such issues considering also if cosmological Noether symmetries in the "point–like" Lagrangian are conformally preserved.

Through this analysis and through also a careful analysis of the cosmological parameters Ω and Λ, it is possible to contribute to the discussion on which is the physical system.

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1 Introduction

Alternative theories of gravity has been formulated and investigated in different context. The Brans–Dicke approach [1], closely related to the Jordan approach [2], has been carried on in the context of a generalization of the Mach's principle; in that approach the Einstein theory of gravitation is modified by introducing a scalar field with a non standard coupling with gravity, *i.e.* the gravitational coupling turns out to be no longer constant. Later on more general couplings have been considered, and the compatibility of such approaches with the different formulations of the Equivalence Principle have been considered [3] [4] [5] [6] [7].

A generalization of the standard gravity comes also from quantum field theories on curved space–times; in such context we find the so called higher order gravitational theories [8] [9] [10].

In all these approaches, the problem of reducing both these two kinds of more general theories in Einstein standard form has been estensively treated; one can see that, through a "Legendre" transformation on the metric, higher order theories, under suitable regularity conditions on the Lagrangian, take the form of the Einstein one in which a scalar field (or more than one) is the source of the gravitational field (see for example [8] [12] [14]); on the other side, it has been studied the equivalence between models with G–variable with the Einstein standard gravity through a suitable conformal transformation (see [3] [4]).

In this paper we analyse, through an appropriately defined conformal transformation, the problem of the equivalence between the non minimally coupled (NMC) theories and the Einstein gravity for scalar–tensor theories in absence of ordinary matter. First, we will do it in the general context and then in the cosmological case, that is, we will study the conformal invariance with the hypotheses of homogeneity and isotropy. In such case we also consider the case in which ordinary matter is present beside the scalar field and we do some consideration on the problem of which is the "physical system" between the two conformally equivalent systems [8] [12] [13].

Furthermore, we analyse the relation between the conformal equivalence and the existence of a Noether symmetry in the $(a,\ \phi)$–space seen as configuration space (*i.e.* in the minisuperspace), where the cosmological "point–like" Lagrangian is defined (we will better clarify the meaning of such expression in our forthcoming considerations); of course such Lagrangian density comes from the general field Lagrangian density once homogeneity and isotropy are assumed [15].

We conclude discussing some examples of physical interest.

2 Conformally equivalent theories

In four dimension, the most general action involving gravity nonminimally coupled with one scalar field is

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[F(\phi)R + \frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V(\phi) \right] \tag{1}$$

where R is the Ricci scalar, $V(\phi)$ and $F(\phi)$ are generic functions describing respectively the potential for the field ϕ and the coupling of ϕ with the gravity; the metric signature is $(+---)$. We use Planck units.

The variation with respect to $g_{\mu\nu}$ gives rise to the field equations

$$F(\phi)G_{\mu\nu} = -\frac{1}{2} T_{\mu\nu} - g_{\mu\nu} \Box_\Gamma F(\phi) + F(\phi)_{;\mu\nu} \tag{2}$$

which are the generalized Einstein equations; here $\Box_\Gamma$ is the d'Alembert operator with respect to the connection Γ;

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{3}$$

is the Einstein tensor, and

$$T_{\mu\nu} = \phi_{;\mu} \phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \phi_{;\alpha} \phi^{\alpha}_{;} + g_{\mu\nu} V(\phi) \tag{4}$$

is the energy–momentum tensor relative to the scalar field.

The variation with respect to ϕ provides the Klein–Gordon equation

$$\Box_\Gamma \phi - R F_\phi(\phi) + V_\phi(\phi) = 0 \tag{5}$$

where $F_\phi = dF(\phi)/d\phi$, $V_\phi = dV(\phi)/d\phi$. This last equation is equivalent to the Bianchi contracted identity (see [15]).

Let us consider now a conformal transformation on the metric $g_{\mu\nu}$ [16], that is

$$\bar{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu} \tag{6}$$

in which $e^{2\omega}$ is the conformal factor. Under this transformation the connection, the Riemann and Ricci tensors, and the Ricci scalar transform in the corresponding way [16], so that the Lagrangian density in (1) becomes

$$\begin{aligned} \sqrt{-g}(FR + \tfrac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V) \; &= \sqrt{-\bar{g}} e^{-2\omega} (F\bar{R} - 6F \Box_{\bar{\Gamma}} \omega + \\ &\quad -6F \omega_{;\alpha} \omega^{\alpha}_{;} + \tfrac{1}{2} \bar{g}^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - e^{-2\omega} V) \end{aligned} \tag{7}$$

in which $\bar{R}$, $\bar{\Gamma}$ and $\Box_{\bar{\Gamma}}$ are respectively the Ricci scalar and the connection relative to the metric $\bar{g}_{\mu\nu}$, and the d'Alembert operator relative to the connection $\bar{\Gamma}$. If we require the theory in the metric $\bar{g}_{\mu\nu}$ to appear as a standard Einstein theory, we get at once that the

conformal factor has to be related to F, that is (see also [4] for a review of particular cases)

$$e^{2\omega} = -2F. \tag{8}$$

We see that F must be negative. Using this relation, the Lagrangian density (7) becomes

$$\sqrt{-g}\left(FR + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V\right) = \sqrt{-\bar{g}}\left(-\frac{1}{2}\bar{R} + 3\Box_{\bar{\Gamma}}\omega + \frac{3{F_\phi}^2 - F}{4F^2}\phi_{;\alpha}\phi^{\alpha}_{;} - \frac{V}{4F^2}\right), \tag{9}$$

Introducing a new scalar field $\bar{\phi}$ and the potential $\bar{V}$, respectively, defined by

$$\bar{\phi}_{;\alpha} = \sqrt{\frac{3{F_\phi}^2 - F}{2F^2}}\,\phi_{;\alpha}, \quad \bar{V}(\bar{\phi}(\phi)) = \frac{V(\phi)}{4F(\phi)} \tag{10}$$

we get

$$\sqrt{-g}\left(FR + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V\right) = \sqrt{-\bar{g}}\left(-\frac{1}{2}\bar{R} + \frac{1}{2}\bar{\phi}_{;\alpha}\bar{\phi}^{\alpha}_{;} - \bar{V}\right), \tag{11}$$

which is the usual Einstein–Hilbert Lagrangian density plus the standard Lagrangian density relative to the scalar field $\bar{\phi}$ (see [15]). (We have not considered the divergence–type term appearing in the Lagrangian (11); we will return on this point in our forthcoming considerations). Therefore, any nonminimally coupled theory, in absence of ordinary matter, is conformally equivalent to an Einstein theory, being the conformal transformation and the potential opportunely defined by (8) and (10) (see also [5]). The converse is also true: for a given $F(\phi)$, such that $3{F_\phi}^2 - F > 0$, we can trasform a standard Einstein theory into a NMC theory. This means that, in principle, if we are able to solve the field equations in the framework of the Einstein theory in presence of a scalar field with a given potential, we should be able to get the solutions for the class of nonminimally coupled theories, assigned by the coupling $F(\phi)$, via the conformal transformation defined by the (8) the only constraint being the second of (10). This is exactly what we are going to discuss in the cosmological context in cases in which the potentials as well as the couplings are relevant from the point of view of the fundamental physics.

Following the standard terminology, we denote here as "Einstein frame" the framework of the Einstein theory, also indicated as minimally coupled theory and as "Jordan frame" the framework of the nonminimally coupled theory [2].

There are some remarks to do with respect to (9) and (10): first we want to stress that the "new" scalar field as defined in (10) is given in differential form in terms of the "old" one and its integration can be not trivial; the second remark concerns the divergence appearing in (9). The transformed Lagrangian density obtained from (7) imposing (8) contains a divergence term, in which appears not only the metric but also its derivative, through the connection $\bar{\Gamma}$. Therefore the equivalence of this total Lagrangian density to the Einstein–Hilbert Lagrangian density plus scalar field is not trivial. To check that they are actually equivalent, let us perform the conformal transformation (6) on the field

Eqs. (2), obtaining

$$\begin{aligned}\bar{G}_{\mu\nu} &= \left(-\frac{1}{2F}+\frac{F_{\phi\phi}}{F}+\frac{2\omega_\phi F_\phi}{F}-2{\omega_\phi}^2-2\omega_{\phi\phi}\right)\phi_{;\mu}\phi_{;\nu}+\\ &+\left(\frac{1}{4F}-\frac{F_{\phi\phi}}{F}+\frac{\omega_\phi F_\phi}{F}-{\omega_\phi}^2+2\omega_{\phi\phi}\right)\bar{g}_{\mu\nu}\phi_{;\alpha}\phi^{\alpha}_{;}+\left(-\frac{F_\phi}{F}+2\omega_\phi\right)\bar{g}_{\mu\nu}\Box_{\bar{\Gamma}}\phi+\\ &+\left(\frac{F_\phi}{F}-2\omega_\phi\right)(\nabla_{\bar{\Gamma}})_\mu(\nabla_{\bar{\Gamma}})_\nu\phi-\frac{1}{2F}e^{-2\omega}\bar{g}_{\mu\nu}V\end{aligned} \tag{12}$$

in which $(\nabla_{\bar{\Gamma}})_\mu$ is the covariant derivative with respect to x^μ relative to the connection $\bar{\Gamma}$. We see, from (12), that if ω satisfies the relation

$$\frac{F_\phi}{F}-2\omega_\phi=0 \tag{13}$$

Eqs. (12) can be rewritten as

$$\bar{G}_{\mu\nu}=\frac{3{F_\phi}^2-F}{2F^2}\phi_{;\mu}\phi_{;\nu}-\bar{g}_{\mu\nu}\frac{3{F_\phi}^2-F}{2F^2}\phi_{;\alpha}\phi^{\alpha}_{;}-\bar{g}_{\mu\nu}\frac{e^{-2\omega}}{2F}V. \tag{14}$$

Then, performing on ϕ the transformation given by (10) and on V the transformation

$$W(\bar{\phi}(\phi))=-\frac{e^{-2\omega(F)}}{2F}V \tag{15}$$

in which $\omega(F)$ satisfies (13), Eq. (14) becomes

$$\bar{G}_{\mu\nu}=\bar{\phi}_{;\mu}\bar{\phi}_{;\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{\phi}_{;\alpha}\bar{\phi}^{\alpha}_{;}-\bar{g}_{\mu\nu}W, \tag{16}$$

which correspond to the Einstein equations in presence of a scalar field $\bar{\phi}$ with potential W. The expression for $\omega(F)$ is easily obtained from (13), that is

$$\omega=\frac{1}{2}\ln F+\omega_0 \tag{17}$$

in which ω_0 is the integration constant. The potential W takes the form

$$W=-\frac{V}{2\alpha F}. \tag{18}$$

Comparing (18) with the second of (10), we see that, fixing $\alpha=-2$, the definition of W coincides with that one of $\bar{V}$. We have then the full compatibility with the Lagrangian approach obtaining for ω the same relation as (8); in this sense we have verified the full equivalence between the NMC and the Einstein–Hilbert Lagrangian density plus scalar field.

Our final remark regards the relations (10): actually, from (9) the relation between $\bar{\phi}_{;\alpha}$ and $\phi_{;\alpha}$ present a $\pm$ sign in front of the square root, which corresponds to have the same or opposite sign in the derivative of ϕ and $\bar{\phi}$ with respect to x_α. What follows is independent of such sign; we will choose then the positive one.

3 The cosmological case

Let us assume now that the spacetime manifold is described by a FRW metric, that is we consider homogeneous and isotropic cosmology. Then the Ricci scalar, has the expression $R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2}\right)$, in which the dot means the derivative with respect to time and κ is the curvature constant. The Lagrangian density (1) takes the form [15]

$$L_t = 6F(\phi)a\dot{a}^2 + 6F_\phi(\phi)a^2\dot{a}\dot{\phi} - 6F(\phi)a\kappa + \frac{1}{2}a^3\dot{\phi}^2 - a^3V(\phi). \tag{19}$$

Expression (19) can be seen as a "point–like" Lagrangian on the configuration space $(a,\ \phi)$ (in this way the meaning of the expression we used in the introduction is clarified). With the subscript t, we mean that the time–coordinate considered is the universal time t: this remark is important for the forthcoming discussion. The Euler–Lagrange equations relative to (19) are then

$$\begin{cases} \dfrac{2\ddot{a}}{a} + \dfrac{\dot{a}^2}{a^2} + \dfrac{2F_\phi\dot{a}\dot{\phi}}{Fa} + \dfrac{F_\phi\ddot{\phi}}{F} + \dfrac{\kappa}{a^2} + \dfrac{F_{\phi\phi}\dot{\phi}^2}{F} - \dfrac{\dot{\phi}^2}{4F} + \dfrac{V}{2F} = 0 \\ \ddot{\phi} + \dfrac{3\dot{a}\dot{\phi}}{a} + \dfrac{6F_\phi\dot{a}^2}{a^2} + \dfrac{6F_\phi\ddot{a}}{a} + \dfrac{6F_\phi\kappa}{a^2} + V_\phi = 0 \end{cases} \tag{20}$$

which correspond to the (generalized) second order Einstein equation and the Klein–Gordon equation in the FRW case. The energy function relative to (19) is

$$\begin{aligned} E_t &= \frac{\partial L_t}{\partial\dot{a}}\dot{a} + \frac{\partial L_t}{\partial\dot{\phi}}\dot{\phi} - L_t = \\ &= 6Fa\dot{a}^2 + 6F_\phi a^2\dot{a}\dot{\phi} + 6Fa\kappa + \tfrac{1}{2}a^3\dot{\phi}^2 + a^3V, \end{aligned} \tag{21}$$

and we see that the first order generalized Einstein equation is equivalent to

$$E_t = 0. \tag{22}$$

Performing the conformal transformation defined by (6), (8), (10) on the FRW metric, one should obtain the corresponding expression for the Lagrangian and the corresponding equations of the Einstein–cosmology from the NMC Lagrangian (19) and from the generalized Einstein and Klein–Gordon equations, respectively. Unfortunately we see that the presence of the conformal factor (8) implies that the transformed line element which is obtained is no longer expressed in the "universal time form". Actually the scale factor of the Einstein theory can be defined as the scale factor of the NMC theory multiplied by the conformal factor, but the time coordinate of the Einstein theory has to be redefined if we require to have an "universal time". Absorbing the conformal factor in the redefinition of time, we obtain the transformation on the time coordinate. Therefore, the

transformation from the Jordan frame to the Einstein frame in the cosmological case is given by

$$\begin{cases} \bar{a} = \sqrt{-2F(\phi)}a \\ \\ \dfrac{d\bar{\phi}}{dt} = \sqrt{\dfrac{3{F_\phi}^2 - F}{2F^2}}\,\dfrac{d\phi}{dt} \\ \\ d\bar{t} = \sqrt{-2F(\phi)}\,dt. \end{cases} \tag{23}$$

From the first and the third of (23) we have that, on the Jordan–frame solutions $a(t)$, $\phi(t)$, we obtain $\bar{a}$ as a function of $\bar{t}$ only; indeed the important thing is the fact that the equations for $\bar{a}$ we will obtain are the standard Einstein equations. The second of (23) corresponds to relation (10) under the given assumption of homogeneity and isotropy. Under transformation (23) we have that

$$\begin{aligned} \frac{1}{\sqrt{-2F}}L_t &= \frac{1}{\sqrt{-2F}}\left(6Fa\dot{a}^2 + 6F_\phi a^2\dot{a}\dot{\phi} - 6Fa\kappa + \frac{1}{2}a^3\dot{\phi}^2 - a^3V\right) = \\ &= -3\bar{a}\dot{\bar{a}}^2 + 3\kappa\bar{a} + \tfrac{1}{2}\bar{a}^3\dot{\bar{\phi}}^2 - \bar{a}^3\bar{V}(\bar{\phi}) = \overline{L}_{\bar{t}} \end{aligned} \tag{24}$$

in which the dot over barred quantities means the derivative with respect to $\bar{t}$; L_t is given by (19) and $\overline{L}_{\bar{t}}$ coincides with the "point–like" Lagrangian obtained from the Einstein–Hilbert action plus a scalar field under the assumption of homogeneity and isotropy. In this way the invariance of the homogeneus and isotropic action under (23) is insured, being L_t and $\overline{L}_{\bar{t}}$ connected by the (24). The same correspondence as (24) exists between the energy function E_t and $\overline{E}_{\bar{t}}$, that is, there is correspondence between the two first order Einstein equations in the two frames. We focus now our attention on the way the Euler–Lagrange equations transform under (23). The Euler–Lagrange equations relative to (24) are the usual second order Einstein equation and Klein–Gordon equation

$$\begin{cases} \dfrac{2\ddot{\bar{a}}}{\bar{a}} + \dfrac{\dot{\bar{a}}^2}{\bar{a}} + \dfrac{\kappa}{\bar{a}^2} + \dfrac{1}{2}\dot{\bar{\phi}}^2 - \bar{V} = 0 \\ \\ \ddot{\bar{\phi}} + \dfrac{3\dot{\bar{a}}\dot{\bar{\phi}}}{\bar{a}} + \bar{V}_{\bar{\phi}} = 0. \end{cases} \tag{25}$$

Under (23) it is straighforward to verify that they become

$$\begin{cases} \dfrac{2\ddot{a}}{a} + \dfrac{\dot{a}^2}{a^2} + \dfrac{2F_\phi\dot{a}\dot{\phi}}{Fa} + \dfrac{F_\phi\ddot{\phi}}{F} + \dfrac{\kappa}{a^2} + \dfrac{F_{\phi\phi}\dot{\phi}^2}{F} - \dfrac{\dot{\phi}^2}{4F} + \dfrac{V}{2F} = 0 \\ \\ \ddot{\phi} + \dfrac{3\dot{a}\dot{\phi}}{a} + \dfrac{6F_\phi F_{\phi\phi} - F_\phi}{3{F_\phi}^2 - F}\dfrac{\dot{\phi}^2}{2} + \dfrac{2F_\phi V}{3{F_\phi}^2 - F} - \dfrac{FV_\phi}{3{F_\phi}^2 - F} = 0 \end{cases} \tag{26}$$

which do not coincide with the Euler–Lagrange equations given by (20). Using the first of (20), the second of (26) can be written as

$$\frac{F-3{F_\phi}^2}{F}\ddot{\phi}+\frac{3(F-3{F_\phi}^2)}{F}\frac{\dot{a}}{a}\dot{\phi}+\frac{F_\phi-6F_{\phi\phi}F_\phi}{F}\frac{\dot{\phi}^2}{2}+\frac{F_\phi\dot{\phi}^2}{4F}-\frac{2F_\phi V}{F}+V_\phi+$$
$$+\frac{3F_\phi\dot{a}^2}{a^2}+\frac{3F_\phi\kappa}{a^2}+\frac{3{F_\phi}^2\dot{a}\dot{\phi}}{a}=0 \tag{27}$$

which becomes, taking into account (21)

$$\frac{F-3{F_\phi}^2}{F}\ddot{\phi}+\frac{3(F-3{F_\phi}^2)}{F}\frac{\dot{a}\dot{\phi}}{a}+\frac{1}{F}\frac{d}{d\phi}(F-3{F_\phi}^2)\frac{\dot{\phi}^2}{2}+\frac{F_\phi\dot{\phi}^2}{4F}-\frac{2F_\phi V}{F}+V_\phi+$$
$$+\frac{F_\phi}{2a^3F}E_t=0. \tag{28}$$

Comparing (28) with the second of (26) we see that they concide if $F-3{F_\phi}^2\neq 0$ and $E_t=0$. The quantity $F-3{F_\phi}^2$ is proportional to the Hessian determinant of L_t with respect to $(\dot{a},\ \dot{\phi})$; we want this Hessian different from zero in order to avoid pathologies in the dynamics [17], while $E_t=0$ corresponds to the first order Einstein equation. It seems that, under the assumption of homogeneity and isotropy and the request of having the metric expressed in the universal time in both the Einstein and Jordan frame, we have conformal equivalence between the Euler–Lagrange Eqs. (20) and (25) only on the (cosmological) solutions. Actually, if we look more carefully to this problem, we notice that, making the hypotheses of homogeneity and isotropy on the field Eqs. (2) and (3), we get the generalized Einstein equations of first and second order, and the Klein–Gordon equation. On the other side, the Euler–Lagrange equations relative to (19) are just the second order Einstein equation and the Klein–Gordon equation, whereas the first order Einstein equation is obtained from $E_t=0$. Of course the same happens in the Einstein frame. Therefore it is natural to expect that the full conformal equivalence in the "point–like" formulation is verified taking into consideration $E_t=0$.

It is possible to see more clearly at the problem of the cosmological conformal equivalence, formulated in the context of the "point–like" Lagrangian, if we use as time–coordinate the conformal time η, connected to the universal time t by the usual relation

$$a^2(\eta)d\eta^2=dt^2. \tag{29}$$

We can see that the use of η makes much easier the treatment of all the problems we have discussed till now.

The crucial point is the following: given the form of the FRW line element expressed in conformal time η one does not face the problem of redefining time after performing a conformal transformation, since in this case, the expansion parameter appears in front of

all the terms of the line element. From this point of view, the conformal transformation which connects Einstein and Jordan frame is given by

$$\begin{cases} \bar{a} = \sqrt{-2F(\phi)}a \\ \\ \dfrac{d\bar{\phi}}{d\eta} = \sqrt{\dfrac{3{F_\phi}^2 - F}{2F^2}}\,\dfrac{d\phi}{d\eta} \end{cases} \tag{30}$$

where a, ϕ, $\bar{a}$, $\bar{\phi}$ are assumed as functions of η.

The Einstein–Hilbert "point–like" Lagrangian is given by

$$\overline{L}_\eta = -3\bar{a}'^2 + 3\kappa\bar{a}^2 + \frac{1}{2}\bar{a}^2\bar{\phi}'^2 - \bar{a}^4\bar{V}(\bar{\phi}) \tag{31}$$

in which the prime means the derivative with respect to η, and the subscript η means that the time–coordinate considered is the conformal time. Under transformation (30), it becomes

$$\begin{aligned} \overline{L}_\eta &= -3\bar{a}'^2 + 3\kappa\bar{a}^2 + \tfrac{1}{2}\bar{a}^2\bar{\phi}'^2 - \bar{a}^4\bar{V}(\bar{\phi}) \\ &= 6F(\phi)a' + 6F_\phi(\phi)aa'\phi' - 6F(\phi)\kappa a^2 + \tfrac{1}{2}a^2\phi'^2 - a^4V(\phi) = L_\eta \end{aligned} \tag{32}$$

which corresponds to the "point–like" Lagrangian obtained from the Lagrangian density in (1) under the hypotheses of homogeneity and isotropy, using the conformal time as time coordinate.

This means that the Euler–Lagrange equations relative to (31), which coincides with the second order Einstein equation and the Klein–Gordon equation in conformal time, correspond to the Euler–Lagrange equations relative to (32), under the transformation (30). Moreover, the energy function $\overline{E}_\eta$ relative to (29) corresponds to the energy function E_η relative to (32), so that there is correspondence between the first order Einstein equations. Furthermore, in order to have full coherence between the two formulations, it is easy to verify that, both in the Jordan frame and in the Einstein frame, the Euler–Lagrange equations, written using the conformal time, correspond to the Euler–Lagrange equations written using the universal time except for terms in the energy function; for it one gets the relation

$$E_\eta = aE_t \tag{33}$$

which holds in both the frames; thus the first order Einstein equation is preserved under the transformation from η to t and there is full equivalence between the two formulations. We want to point out that for the two Lagrangians L_η and L_t the same relation as (33) holds; this remark is useful for forthcoming considerations.

4 The presence of ordinary matter

So far we have analysed the general conformal equivalence and the cosmological conformal equivalence between Einstein frame and Jordan frame in presence of a scalar field.

What happens when ordinary matter is present (see [3])? We focus our attention on the cosmological case.

The standard Einstein (cosmological) "point–like" Lagrangian (when noninteracting scalar field and ordinary matter are present) is given by

$$\overline{L}_{tot} = \overline{L}_{\bar{t}} + \overline{L}_{mat} \tag{34}$$

in which $\overline{L}_{\bar{t}}$ is given by (24) and $\overline{L}_{mat}$ is the Lagrangian relative to matter. Using the contracted Bianchi identity, it can be seen that $\overline{L}_{mat}$ can be written as [19]

$$\overline{L}_{mat} = -D\bar{a}^{3(1-\gamma)} \tag{35}$$

where D is connected with the total amount of matter. In writing (34) and (35) we have chosen the universal time as time–coordinate. Under the transformation (23) we have, beside relation (24), that (35) corresponds to

$$\overline{L}_{mat} = (\sqrt{-2F})^{3(1-\gamma)} L_{mat} \tag{36}$$

where, analogously to (35)

$$L_{mat} = Da^{3(1-\gamma)}. \tag{37}$$

Then we have that, under (23), (34) becomes

$$\frac{1}{\sqrt{-2F}} L^{(1)}_{tot} = \frac{1}{\sqrt{-2F}} [L_t + (\sqrt{-2F})^{(4-3\gamma)} L_{mat}] \tag{38}$$

in which we have defined the total "point–like" Lagrangian after the conformal transformation as

$$L^{(1)}_{tot} = L_t + (\sqrt{-2F})^{(4-3\gamma)} L_{mat}, \tag{39}$$

(cfr. (24)); the transformation of $\overline{L}_{tot}$ under (23) has to be written following the expression (38) and consequently the "point–like" Lagrangian $L^{(1)}_{tot}$, has to be defined as in (39). The use of the superscript (1) for L_{tot} will be clarified in a moment. The factor $\frac{1}{\sqrt{-2F}}$ in evidence out of the square bracket, is introduced in order to preserve the invariance of the reduced action under transformation (23), since that factor is also the one which appears in the time–coordinate transformation in (23).

The Lagrangian (39) could be then assumed to describe a cosmological NMC–model with a scalar field and ordinary matter as gravitational sources. By the way, we see that, unless $\gamma = \frac{4}{3}$, the standard matter Lagrangian term is coupled with the scalar field in a way which depends on the coupling F. Such coupling between the matter and the scalar field is an effect of the transformation, therefore depending on the coupling F. This is one way to look at the problem, but we can also proceed in a different way to determine the Lagrangian in presence of matter. We can consider as the total "point–like" Lagrangian

$$L^{(2)}_{tot} = L_t + L_{mat}. \tag{40}$$

That is, we take the "point–like" Lagrangian of the NMC theories, given by (19), and add up to it the standard matter term defined in (37). It is clear now why we have introduced the notation $L^{(1)}_{tot}$ and $L^{(2)}_{tot}$. Of course the full theory described by (40) is by no mean conformally equivalent to that one described by (34). Also, the transformation does not give rise to any coupling between matter and scalar field. One could just point out that the matter term defined by (37) has been obtained in the context of the Einstein frame and in this sense it could be not legitimate using it in a NMC theory.

The problem of the physical system, also connected with the formulation of the Equivalence Principle [20], has been already discussed and it is well known, in particular in the case of the higher order theories [11] [12] [13] [14] (and references quoted therein), but also in the context of NMC theories [3] [5]. In the case we are considering, the problem still concerns the choice of the physical system but from another point of view, since the Lagrangians (39) and (40) are not connected by a conformal transformation. The problem concerns which is the Lagrangian to describe, in the Jordan frame, a cosmological model with a scalar field plus ordinary matter, between the lagrangian $L^{(1)}_{tot}$ and $L^{(2)}_{tot}$, once we assume that the physical system is that one of a NMC theory (some authors consider the Jordan frame as the physical one, see for examples [21], while in [22] the Einstein frame is the physical one).

This sort of ambiguity can be clarified in the general context of the field theory, introduced in Sec. 2. We focus our attention on the contracted Bianchi identity. From the point of view of the field equations, the choice of the Lagrangian $L^{(2)}_{tot}$ to describe the gravitational field with a scalar field and ordinary matter (non interacting with the scalar field) as sources in the Jordan frame corresponds to write the field equations as

$$F(\phi)G_{\mu\nu} = -\frac{1}{2}T_{\mu\nu} - g_{\mu\nu}\Box_\Gamma F(\phi) + F(\phi)_{;\mu\nu} + T_{(mat)\ \mu\nu} \tag{41}$$

which is obtained just adding up the ordinary matter as a further source term to to the field equations in the NMC case given by (2). Performing the covariant divergence of both sides and taking into account of the expression of $T_{(\phi)\ \mu\nu}$ given by (4), we get

$$F_{;\nu}G_{\mu}^{\ \nu} + \frac{1}{2}\phi_{;\mu}\phi^{\ \ \mu}_{;\ ;\nu} + \frac{1}{2}\delta_\mu^{\ \nu}V_\phi\phi_{;\nu} + \delta_\mu^{\ \nu}(\Box_\Gamma F)_{;\nu} - F_{;\mu}^{\ \ \nu}{}_{\nu} = T_{(mat)\ \mu\ ;\nu}^{\ \ \ \nu} \tag{42}$$

which can be written as

$$F_{;\nu}R_{\mu}^{\ \nu} - \frac{1}{2}RF_{;\mu} + \frac{1}{2}\phi_{;\mu}(\Box_\Gamma\phi + V_\phi) + (\Box_\Gamma F)_{;\mu} - F_{;\mu}^{\ \ \nu}{}_{\nu} = T_{(mat)\ \mu\ ;\nu}^{\ \ \ \nu} \tag{43}$$

where we have taken into account Eq. (3). The last two terms of the lefthand side of (43) give

$$\begin{aligned}(g^{\alpha\beta}F_{;\alpha\beta})_{;\mu} - F_{;\mu}^{\ \ \nu}{}_{\nu} &= -g^{\alpha\beta}R^{\lambda}{}_{\beta\alpha\nu}F_{;\lambda} \\ &= -F_{;\lambda}R_{\mu}^{\ \lambda}.\end{aligned} \tag{44}$$

Eq. (43), taking into account Eq. (44) gives then an interesting relation, that is

$$\frac{1}{2}\phi_{;\mu}(\Box_\Gamma\phi + V_\phi - RF_\phi) = T_{(mat)\ \mu}^{\ \ \ \nu}{}_{;\nu}. \tag{45}$$

Comparing relation (45) with Eq. (5) we see that the lefthand side coincides with the lefthand side of the Klein–Gordon equation; it means that the continuity equation for the ordinary matter holds, that is

$$T_{(mat)\ \mu}^{\ \ \ \nu}{}_{;\nu} = 0 \tag{46}$$

in which $T_{(mat)\ \mu}^{\ \ \ \nu}{}_{;\nu}$ is the energy–momentum tensor of the matter, relative to the NMC metric $g_{\mu\nu}$. Thus, it means that, choosing the Jordan frame as the physical frame and equations (41) as field equations, the conservation of matter is relative to the physical metric $g_{\mu\nu}$. In this sense the legitimate way to describe scalar field plus ordinary matter in the Jordan frame is the one given by the field equations (41). The corresponding action, in the cosmological case, corresponds to choose $L^{(2)}_{tot}$ as Lagrangian [11] [12] [13] [14] (and ref. quoted therein).

We can only say that such considerations could be a hint for further developments in the context of the Jordan frame (for a totally different point of view see [5]).

The problem can be further analysed from the point of view of the energy density parameter Ω. We can see in fact that the presence of the coupling gives some contributions to Ω. Let us consider the first order Einstein equation relative to the total lagrangian $L^{(2)}_{tot}$ in the Jordan frame in presence of matter

$$\frac{\dot{a}^2}{a^2} + \frac{F_\phi \dot{a}\dot{\phi}}{Fa} + \frac{\kappa}{a^2} + \frac{1}{2}\frac{\dot{\phi}^2}{6F} + \frac{V}{6F} + \frac{D}{6Fa^{3\gamma}} = 0 \tag{47}$$

which can be seen as obtained from the standard first order Einstein equation after the conformal transformation (23), having just added ordinary matter. The last term on the lefthand side being just the effective energy density relative to matter; the factor $-\frac{1}{2F}$ represents the effective coupling. Taking into account the definition of the Hubble parameter H, (47) can be rewritten as

$$H^2 + \frac{\dot{F}H}{F} + \frac{1}{6F}\left(\frac{1}{2}\dot{\phi}^2 + V\right) + \frac{D}{6Fa^{3\gamma}} = 0 \tag{48}$$

(we consider the case $\kappa = 0$). We get then

$$-\frac{\dot{F}}{FH} - \frac{1}{6H^2F}\left(\frac{1}{2}\dot{\phi}^2 + V\right) - \frac{D}{6H^2Fa^{3\gamma}} = 1 \tag{49}$$

after dividing by H^2. As usual, the righthand side is the total energy density parameter Ω_{tot} which is equal to 1, having assumed $\kappa = 0$. The last term on the lefthand side represents the effective contribution to the density parameter due to the matter, $\Omega_{mat} =$

$-\frac{D}{6H^2Fa^{3\gamma}}$, while the term in parentesis represents the effective energy density contribution due to the scalar field, $\Omega_\phi = -\frac{1}{6H^2F}\left(\frac{1}{2}\dot{\phi}^2 + V\right)$; the first term is connected with the variation of the coupling, $\Omega_{coup} = -\frac{\dot{F}}{FH}$. That is, there is a contribution to the energy density parameter due to the presence of the nonminimal coupling. This term, for what we have said, can be seen as coming from the conformal transformation considered. The parameter Ω is an observative quantity (the present value of Ω_{tot} is assumed to be equal to 1) thus, in principle, from its analysis one could be able to infer whether the physical frame is the Jordan or the Einstein frame (see also [12]).

A final remark we would like to do concerns the case $\gamma = \frac{4}{3}$. As we have said, performing the transformation (23) on (34) no coupling between the scalar field and the matter is induced if the matter is a radiative perfect fluid: this seems to be quite reasonable, since the particles which constitute a radiative fluid have zero mass.

5 Conformal transformations and Noether symmetries

We want to analyse now the compatibility between the conformal transformation we have considered so far and the presence of Noether symmetries in the "point–like" Lagrangian in the configuration space $(a,\ \phi)$, *i.e.* in the cosmological case. Some of the authors, in previous papers (see for examples [17] [23]) have developed a method to find exact cosmological solutions relative either to purely scalar–tensor Lagrangians or to scalar–tensor Lagrangians with ordinary matter, both in MC theories, when the Lagrangian is given by (34), and in NMC theories, having taken (40) as Lagrangian.

Now we want to analyse the problem whether the conformal transformation connecting Einstein and Jordan frame preserves the presence of a Noether symmetry. Since the existence of a Noether symmetry implies the existence of a vector field X along which $\mathcal{L}_X L = 0$, this happens if the Lie derivative of the Lagrangian along a vector field is preserved. We can see that the Lie derivative is preserved under the conformal transformation considered, but only in absence of ordinary matter. It turns out to be quite simple to be verified if we choose as time–coordinate the conformal time. As we have seen, we have that in absence of matter, using the time η the "point–like" Lagrangian in the Einstein and Jordan frame given by (31) and (32) respectively, correspond to each other under the conformal transformation given by (30). The second of (30), in principle, can be integrated, so that its finite form together with the first of (30) can represent a "coordinate transformation" on the configuration space $(a,\ \phi)$. Thus, a given lift–vector field of the form [24]

$$X_\eta = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \alpha'\frac{\partial}{\partial a'} + \beta'\frac{\partial}{\partial \phi'} \tag{50}$$

in which $\alpha = \alpha(a,\ \phi)$, $\beta = \beta(a,\ \phi)$ corresponds under this transformation to the lift–vector field on the configuration space $(\bar{a},\ \bar{\phi})$

$$\bar{X}_\eta = \bar{\alpha}\frac{\partial}{\partial \bar{a}} + \bar{\beta}\frac{\partial}{\partial \bar{\phi}} + \bar{\alpha}'\frac{\partial}{\partial \bar{a}'} + \bar{\beta}'\frac{\partial}{\partial \bar{\phi}'} \tag{51}$$

in which $\bar{\alpha} = \bar{\alpha}(\bar{a},\ \bar{\phi})$, $\bar{\beta} = \bar{\beta}(\bar{a},\ \bar{\phi})$ are connected to $\alpha = \alpha(a,\ \phi)$, $\beta = \beta(a,\ \phi)$ through the Jacobian matrix relative to the "coordinate transformation" defined by (30). We remind that the prime means the derivative with respect to the time η. The Lie derivative of L_η along the vector field X_η corresponds then to the Lie derivative of $\overline{L}_\eta$ along $\bar{X}_\eta$ [25]

$$\mathcal{L}_{X_\eta} L_\eta = \mathcal{L}_{\bar{X}_\eta} \overline{L}_\eta. \tag{52}$$

Therefore, if X_η is a Noether vector field relative to L_η one has

$$\mathcal{L}_{X_\eta} L_\eta = 0 \tag{53}$$

and, from (52), $\bar{X}_\eta$ is a Noether vector field relative to $\overline{L}_\eta$.

We have seen till now that the choice of η as time–coordinate is convenient from a formal point of view, but, as we have already remarked, in order to analyse the phenomenology relative to a given model and to obtain then quantities comparable with the observational data, the appropriate choice of time–coordinate is the universal time t. The problem with the universal time is that it is not preserved by the conformal transformation, as we have pointed out in Sec. 2, thus the conformal transformation we consider does not take simply the form of a "coordinate transformation" on the phase space $(a,\ \phi)$, then its compatibility with the presence of a Noether symmetry cannot be easily verified. Of course it must hold also under such choice of time–coordinate.

We decide not to verify such compatibility directly. Rather, we analyse how does the Lie derivative $\mathcal{L}_{X_\eta} L_\eta$ in the Jordan frame is transformed under the time transformation (29) which connects t with η.

The explicit expression of $\mathcal{L}_{X_\eta} L_\eta$ is given by

$$\begin{aligned}
\mathcal{L}_{X_\eta} L_\eta \ &= 6\left[2F\frac{\partial\alpha}{\partial a} + \left(\beta + a\frac{\partial\beta}{\partial a}\right)F_\phi\right]a'^2 + a\left[\alpha + 6F_\phi\frac{\partial\alpha}{\partial\phi} + a\frac{\partial\beta}{\partial\phi}\right]\phi'^2 + \\
&+6\left[a\beta F_{\phi\phi} + \left(\alpha + a\frac{\partial\alpha}{\partial a} + a\frac{\partial\beta}{\partial\phi}\right)F_\phi + 2F\frac{\partial\alpha}{\partial\phi} + \frac{a^2}{6}\frac{\partial\beta}{\partial a}\right]a'\phi' + \\
&-a^3(4\alpha V + a\beta V_\phi) - 6a(2F\alpha + F_\phi a\beta)\kappa
\end{aligned} \tag{54}$$

in which we have taken into account that

$$\alpha' = \frac{\partial\alpha}{\partial a}a' + \frac{\partial\alpha}{\partial\phi}\phi'; \quad \beta' = \frac{\partial\beta}{\partial a}a' + \frac{\partial\beta}{\partial\phi}\phi'; \tag{55}$$

(54) under the transformation (29) becomes

$$\begin{aligned}
\mathcal{L}_{X_\eta} L_\eta \ &= 6\left[2F\frac{\partial\alpha}{\partial a} + \left(\beta + a\frac{\partial\beta}{\partial a}\right)F_\phi\right]a^2\dot{a}^2 + a\left[\alpha + 6F_\phi\frac{\partial\alpha}{\partial\phi} + a\frac{\partial\beta}{\partial\phi}\right]a^2\dot{\phi}^2 + \\
&+6\left[a\beta F_{\phi\phi} + \left(\alpha + a\frac{\partial\alpha}{\partial a} + a\frac{\partial\beta}{\partial\phi}\right)F_\phi + 2F\frac{\partial\alpha}{\partial\phi} + \frac{a^2}{6}\frac{\partial\beta}{\partial a}\right]a^2\dot{a}\dot{\phi} + \\
&-a^3(4\alpha V + a\beta V_\phi) - 6a(2F\alpha + F_\phi a\beta)\kappa
\end{aligned} \tag{56}$$

which can be written as

$$\begin{aligned}\mathcal{L}_{X_\eta}L_\eta &= 6a\left[\alpha F+2aF\frac{\partial\alpha}{\partial a}+a\left(\beta+a\frac{\partial\beta}{\partial a}\right)F_\phi\right]\dot{a}^2+a^3\left[\frac{3}{2}\alpha+6F_\phi\frac{\partial\alpha}{\partial\phi}+a\frac{\partial\beta}{\partial\phi}\right]\dot{\phi}^2+\\ &+6a^2\left[a\beta F_{\phi\phi}+\left(2\alpha+a\frac{\partial\alpha}{\partial a}+a\frac{\partial\beta}{\partial\phi}\right)F_\phi+2F\frac{\partial\alpha}{\partial\phi}+\frac{a^2}{6}\frac{\partial\beta}{\partial a}\right]\dot{a}\dot{\phi}+\\ &-a^3(3\alpha V+a\beta V_\phi)-6a(F\alpha+F_\phi a\beta)\kappa+\\ &-6\alpha Fa\dot{a}^2-\frac{1}{2}\alpha a^3\phi^2-6\alpha F_\phi a^2\dot{a}\dot{\phi}-6\alpha Fa\kappa-\alpha a^3V.\end{aligned} \tag{57}$$

The Lie derivative of L_t given by (19) along a lift–vector field of the form

$$X_t=\alpha\frac{\partial}{\partial a}+\beta\frac{\partial}{\partial\phi}+\dot{\alpha}\frac{\partial}{\partial\dot{a}}+\bar{\beta}\frac{\partial}{\partial\dot{\phi}} \tag{58}$$

is given by

$$\begin{aligned}\mathcal{L}_{X_t}L_t &= 6\left[\alpha F+2aF\frac{\partial\alpha}{\partial a}+a\left(\beta+a\frac{\partial\beta}{\partial a}\right)F_\phi\right]\dot{a}^2+a^2\left[\frac{3}{2}\alpha+6F_\phi\frac{\partial\alpha}{\partial\phi}+a\frac{\partial\beta}{\partial\phi}\right]\dot{\phi}^2+\\ &+6a\left[a\beta F_{\phi\phi}+\left(2\alpha+a\frac{\partial\alpha}{\partial a}+a\frac{\partial\beta}{\partial\phi}\right)F_\phi+2F\frac{\partial\alpha}{\partial\phi}+\frac{a^2}{6}\frac{\partial\beta}{\partial a}\right]\dot{a}\dot{\phi}+\\ &-a^2(3\alpha V+a\beta V_\phi)-6(F\alpha+F_\phi a\beta)\kappa\end{aligned} \tag{59}$$

in which we have taken into account that

$$\dot{\alpha}=\frac{\partial\alpha}{\partial a}\dot{a}+\frac{\partial\alpha}{\partial\phi}\dot{\phi};\quad \dot{\beta}=\frac{\partial\beta}{\partial a}\dot{a}+\frac{\partial\beta}{\partial\phi}\dot{\phi}. \tag{60}$$

We remind that the dot means the derivative with respect to t.

Comparing (57) with (59) and taking into account the expression of E_t given by (21) we obtain that, under the transformation (29) the Lie derivative $\mathcal{L}_{X_\eta}L_\eta$ becomes

$$\mathcal{L}_{X_\eta}L_\eta=a\mathcal{L}_{X_t}L_t-(\mathcal{L}_{X_t}a)E_t, \tag{61}$$

being $\mathcal{L}_{X_t}a=\alpha$.

It can be seen that the same relation as (61) holds in the Einstein frame, that is

$$\mathcal{L}_{\bar{X}_\eta}\overline{L}_\eta=\bar{a}\mathcal{L}_{\bar{X}_{\bar{t}}}\overline{L}_{\bar{t}}-(\mathcal{L}_{\bar{X}_{\bar{t}}}\bar{a})\overline{E}_{\bar{t}}, \tag{62}$$

with quite obvious meaning of $\bar{X}_{\bar{t}}$.

This implies that, if X_η is a Noether vector field relative to L_η, that is, if (53) holds, the corresponding vector field X_t is such that

$$\mathcal{L}_{X_t} L_t - \frac{\mathcal{L}_{X_t} a}{a} E_t = 0. \tag{63}$$

It means also that, when the universal time is taken as time–coordinate, the conformal transformation preserves the expression given by the righthand side of (61) and not the Lie derivative along a given vector field X_t. Relation (63) represents a more general way to express the presence of a first integral for the Lagrangian L_t; associated to (63) we have the conserved quantity [26]

$$- E_t \int \frac{\mathcal{L}_{X_t}}{a} dt + \frac{\partial L_t}{\partial \dot{a}} \alpha + \frac{\partial L_t}{\partial \dot{\phi}} \beta = const \tag{64}$$

which, of course, holds on the solutions of the Euler–Lagrange equations. The vector field X_t verifying (63) can thus be seen as a generalized Noether vector field and the conformal transformation (23) preserves this generalized symmetry. That is, if X_t is a Noether vector field, in the sense of (63), relative to L_t then $\bar{X}_t$ is a Noether vector field relative to $\overline{L}_{\bar{t}}$ in the same sense, that is

$$\mathcal{L}_{\bar{X}_{\bar{t}}} \overline{L}_{\bar{t}} - \frac{\mathcal{L}_{\bar{X}_t} \bar{a}}{\bar{a}} \overline{E}_t = 0. \tag{65}$$

In terms of the conformal time, the first integral relative to (63) for the Lagrangian L_η is given by

$$\frac{\partial L_\eta}{\partial a'} \alpha + \frac{\partial L_\eta}{\partial \phi'} \beta = const. \tag{66}$$

We see that the expression (66) corresponds to (64) under the transformation (29), except for a term in the energy function. In fact (66) explicitely written is

$$(12Fa' + 6F_\phi a\phi')\alpha + (6F_\phi aa' + a^2\phi')\beta = const, \tag{67}$$

while (64) is

$$- E_t \int \frac{\alpha}{a} dt + (12Fa\dot{a} + 6F_\phi a^2 \dot{\phi})\alpha + (6F_\phi a^2 \dot{a} + a^3 \dot{\phi})\beta = const. \tag{68}$$

Taking into account (33), we have that (68), under (29), becomes

$$- \frac{E_\eta}{a} \int \alpha d\eta + (12Fa' + 6F_\phi a\phi')\alpha + (6F_\phi aa' + a\phi')\beta = const \tag{69}$$

which coincides with (67) except for the term in E_η.

Therefore there is equivalence between the two formulations except for the term in E_η, coherently with what we have said at the end of Sec. 2.

As already said, some of the authors have formulated the existence of a Noether vector field imposing

$$\mathcal{L}_{X_t} L_t = 0 \tag{70}$$

using the universal time as time–coordinate; condition (70) after the analysis we have done till now, turns out to be less general than (65). By the way, condition (70) has the interesting property that it implies the possibility to define some new coordinates on the configuration space $(a,\ \phi)$, such that the Lagrangian has a cyclic coordinate [15] [24], reducing in this way the Euler–Lagrange equations. In fact, one can always define new coordinate, say $(z,\ w)$, in the configuration space of the Lagrangian, such that the lift–vector field assumes the form $X_t = \frac{\partial}{\partial z}$, so that one has $\mathcal{L}_{X_t} L = \frac{\partial L}{\partial z}$; in case (70) holds one has then that z is cyclic. In the generalized case we are considering, it is no longer possible to get this behavior, since $\mathcal{L}_{X_t} L_t \neq 0$ and consequently z is no longer cyclic. In this case one has to use the first integral (64) together with the relation on the energy function to reduce the Euler–Lagrange equations, that is, one has the system of Eqs. (20), Eq. (22) and Eq. (64).

This problem corresponds in the Einstein frame to the system of equations (25), the equation analogue to (22), $\overline{E}_{\bar{t}} = 0$, and the equation analogue to (64),

$$-\overline{E}_{\bar{t}} \int \frac{\mathcal{L}_{\bar{X}_{\bar{t}}}}{\bar{a}} d\bar{t} + \frac{\partial \overline{L}_{\bar{t}}}{\partial \dot{\bar{a}}} \bar{\alpha} + \frac{\partial \overline{L}_{\bar{t}}}{\partial \dot{\bar{\phi}}} \bar{\beta} = const. \tag{71}$$

Thus, finding the solutions of some cosmological model using the presence of a Noether symmetry (and therefore fixing the class of model compatible with it) in the Einstein frame, one gets via the conformal transformation (as given by (23)) the solutions to the class of models in the Jordan frame corresponding to the one given in the Einstein frame through the second of (10). We are going to give some significant examples in the following section.

6 Examples

i) Let us consider a quite easily solvable model in the Einstein frame. We consider the cosmological model with a scalar field, a constant potential and zero curvature. The Lagrangian is given by

$$\overline{L}_{\bar{t}} = -3\bar{a}\dot{\bar{a}}^2 + \frac{1}{2}\bar{a}^3 \dot{\bar{\phi}}^2 - \bar{a}^3 \Lambda; \tag{72}$$

the Euler–Lagrange equations and the zero energy function condition are given by

$$\begin{cases} \dfrac{2\ddot{\bar{a}}}{\bar{a}} + \dfrac{\dot{\bar{a}}^2}{\bar{a}} + \dfrac{1}{2}\dot{\bar{\phi}}^2 - \Lambda = 0 \\ \\ \ddot{\bar{\phi}} + \dfrac{3\dot{\bar{a}}\dot{\bar{\phi}}}{\bar{a}} = 0. \end{cases} \tag{73}$$

$$\frac{\dot{\bar{a}}^2}{\bar{a}^2} - \frac{1}{3}\left(\frac{1}{2}\dot{\bar{\phi}}^2 + \Lambda\right) = 0. \tag{74}$$

The solutions are (see also [27])

$$\begin{cases} \bar{a} = \left[c_1 e^{\sqrt{3\Lambda}\,\bar{t}} - \dfrac{\dot{\bar{\phi}}_0^2}{8\Lambda c_1^2} e^{-\sqrt{3\Lambda}\,\bar{t}}\right]^{\frac{1}{3}} \\ \\ \bar{\phi} = \bar{\phi}_0 + \sqrt{\dfrac{2}{3}}\,\ln \dfrac{1 - \dfrac{\dot{\bar{\phi}}_0}{2c_1\sqrt{2\Lambda}} e^{-\sqrt{3\Lambda}\,\bar{t}}}{1 + \dfrac{\dot{\bar{\phi}}_0}{2c_1\sqrt{2\Lambda}} e^{-\sqrt{3\Lambda}\,\bar{t}}} \end{cases} \tag{75}$$

Of course only three constants of integration appear in the solution, since Eq. (74) corresponds to a contraint on the value of the first integral $\overline{E}_{\bar{t}}$. We have that, in the limit of $\bar{t} \to +\infty$, the behavior of $\bar{a}$ is exponential with characteristic time given by $\sqrt{\frac{\Lambda}{3}}$, as we would expect (see also [18]), and $\bar{\phi}$ goes to a constant.

Looking at the second of (10) we have that such a model in the Einstein frame corresponds in the, Jordan frame, to the class of models with (arbitrarly given) coupling F and potential V connected by the relation

$$\frac{V}{4F^2} = \Lambda, \tag{76}$$

the solution of which can be obtained from (75) via the transformation (23). We can thus fix the potential V and obtain from (76) the corresponding coupling. This can be used as a method to find the solutions of NMC models with given potentials, the coupling being determined by (76). We consider, as an example, the case

$$V = \lambda\phi^4, \quad \lambda > 0 \tag{77}$$

which correspond to a "chaotic inflationary" potential [28]. The corresponding coupling is quadratic in ϕ

$$F = k_0\phi^2 \tag{78}$$

in which

$$k_0 = -\frac{1}{2}\sqrt{\frac{\lambda}{\Lambda}} \tag{79}$$

Substituting (78) into (23) we get

$$\begin{cases} a = \dfrac{\bar{a}}{\phi\sqrt{-2k_0}} \\ d\phi = \phi\sqrt{\dfrac{2k_0}{12k_0-1}}\, d\phi \\ dt = \dfrac{d\bar{t}}{\phi\sqrt{-2k_0}}. \end{cases} \tag{80}$$

As we see from these relations, it has to be $k_0 < 0$. Integrating the second of (80) we have ϕ in terms of $\bar{\phi}$

$$\phi = \alpha_0 e^{\sqrt{\frac{2k_0}{12k_0-1}}\,\bar{\phi}}. \tag{81}$$

Substituting (81) in the first of (80) and taking into account the second of (75) we have the solutions a and ϕ as functions of $\bar{t}$

$$\begin{cases} \phi = \phi_0 \left[\dfrac{1 - \dfrac{\dot{\bar{\phi}}_0}{2c_1\sqrt{2\Lambda}} e^{-\sqrt{3\Lambda}\,\bar{t}}}{1 + \dfrac{\dot{\bar{\phi}}_0}{2c_1\sqrt{2\Lambda}} e^{-\sqrt{3\Lambda}\,\bar{t}}} \right]^{\sqrt{\frac{4k_0}{3(12k_0-1)}}} \\ a = \dfrac{1}{\phi_0\sqrt{-2k_0}} \left[c_1 e^{\sqrt{3\Lambda}\,\bar{t}} - \dfrac{\dot{\bar{\phi}}_0^2}{8\Lambda c_1^2} e^{-\sqrt{3\Lambda}\,\bar{t}} \right]^{\frac{1}{3}} \left[\dfrac{1 + \dfrac{\dot{\bar{\phi}}_0}{2c_1\sqrt{2\Lambda}} e^{-\sqrt{3\Lambda}\,\bar{t}}}{1 - \dfrac{\dot{\bar{\phi}}_0}{2c_1\sqrt{2\Lambda}} e^{-\sqrt{3\Lambda}\,\bar{t}}} \right]^{\sqrt{\frac{4k_0}{3(12k_0-1)}}} \end{cases} \tag{82}$$

in which $\phi_0 = \alpha_0 e^{\sqrt{\frac{2k_0}{12k_0-1}}\,\bar{\phi}_0}$. Substituting (81) in the third of (80), taking into account of (75), we get

$$dt = \frac{d\bar{t}}{\phi_0\sqrt{-2k_0}} \left[\frac{1 + \dfrac{\dot{\bar{\phi}}_0}{2c_1\sqrt{2\Lambda}} e^{-\sqrt{3\Lambda}\,\bar{t}}}{1 - \dfrac{\dot{\bar{\phi}}_0}{2c_1\sqrt{2\Lambda}} e^{-\sqrt{3\Lambda}\,\bar{t}}} \right]^{\sqrt{\frac{4k_0}{3(12k_0-1)}}}. \tag{83}$$

We can obtain $\bar{t}$ as a function of t integrating (83) and then considering the inverse function; (83) could be easily integrated if the exponent $\sqrt{\frac{4k_0}{3(12k_0-1)}}$ would be equal to ± 1, but this corresponds to a value of $k_0 = \frac{3}{32}$ which is positive and thus it turns out to be not acceptable. In general, (83) is not of easy solution. We can analyse its asymptotic

behavior, obtaining

$$\frac{dt}{d\bar{t}} \overset{\bar{t}\to+\infty}{\to} \frac{1}{\phi_0\sqrt{-2k_0}} \tag{84}$$

that is, asymptotically

$$t - t_0 \simeq \frac{\bar{t}}{\phi_0\sqrt{-2k_0}}. \tag{85}$$

Substituting (85) in the asymptotic expression of (82), we obtain the asymptotic behavior of the solutions (since from (84) one has $t \overset{\bar{t}\to+\infty}{\to} +\infty$)

$$\begin{cases} a \simeq \dfrac{c_1^{1/3}}{\phi_0\sqrt{-2k_0}} e^{\phi_0\sqrt{\frac{-2\Lambda k_0}{3}}\,(t-t_0)} \\ \\ \phi \simeq \phi_0. \end{cases} \tag{86}$$

Thus we have that, asymptotically, $a(t)$ is exponential as it had to be (cfr. [18]), and $\phi(t)$ is constant; the coupling F is asymptotically constant too, so that, fixing the arbitrary constant of integration to obtain the finite transformation of $\bar{a}$, $\bar{\phi}$ (that is, fixing the units, see [12]), once k_0 is fixed, it is possible to recover asymptotically the Einstein gravity.

As a remark we would like to notice that the asymptotic expression (86) of $a(t)$ and $\phi(t)$ are solutions of the Einstein equations and Klein–Gordon equation with zero curvature and F given by (77), (78). They have not been obtained as solutions of the asymptotic limits of these equations. It means then that they are, in any case, particular solutions of the given NMC–model.

ii) Another interesting case is the Ginzburg–Landau potential

$$V = \lambda(\phi^2 - \mu^2)^2, \quad \lambda > 0. \tag{87}$$

The corresponding coupling is given by

$$F = k_0(\phi^2 - \mu^2) \tag{88}$$

in which k_0 is given by (79) when $\phi^2 > \mu^2$ while is given by (79) with opposite sign when $\phi^2 < \mu^2$, in order to have $F < 0$. With this coupling the corresponding conformal transformation turns out to be singular for $\phi^2 = \mu^2$, thus with this method it is not possible to solve this model for ϕ equal to the Ginzburg–Landau mass μ.

In this case, it is not so straightforward to get the explicit function $\phi = \phi(\bar{\phi})$ as in the previous case, since one should perform and then obtain the inverse of the integral

$$\bar{\phi} - \bar{\phi}_0 = \int \frac{[3\sqrt{\frac{\lambda}{\Lambda}}\phi^2 + \frac{1}{2}(\phi^2 - \mu^2)]^{\frac{1}{2}}}{[\frac{\lambda}{4\Lambda}]^{\frac{1}{4}}(\phi^2 - \mu^2)}\, d\phi. \tag{89}$$

It is not so difficult to integrate (89) [29], the difficulty raises in finding the inverse, which is needed to obtain t in terms of $\bar{t}$. By the way, analysing the integrand of (89), that is

$\frac{d\bar{\phi}}{d\phi}$, we can say that, except for $\phi^2 = \mu^2$, the function $\bar{\phi} = \bar{\phi}(\phi)$ is invertible; in particular, since asymptotically $\bar{\phi}$ is constant, so is ϕ. Thus it is possible to carry out a reasoning analogous to the previous one, concluding that asymptotically the behavior of $a(t)$ is exponential and that of $\phi(t)$ is constant.

iii) We want to consider now the $V = \lambda\phi^4$ case from the point of view of the Noether symmetries. We want to show that, in the context of generalized Noether symmetries the NMC–model with quartic potential and negative quadratic coupling admits a Noether symmetry, while such a result has been not found in the previous analysis of Noether symmetries (see [15] [17]).

We have seen that the corresponding case in the Einstein frame is that one of constant potential. The system of equations for the Noether vector field obtained from (65) is given by

$$\begin{cases} \dfrac{\partial\bar{\alpha}}{\partial\bar{a}} = 0 \\ \\ \bar{\alpha} + \bar{a}\dfrac{\partial\bar{\beta}}{\partial\bar{\phi}} = 0 \\ 6\dfrac{\partial\bar{\alpha}}{\partial\bar{\phi}} - \bar{a}^2\dfrac{\partial\bar{\beta}}{\partial\bar{a}} = 0 \\ \\ 4\alpha\bar{V} + \bar{a}\bar{\beta}\bar{V}_{\bar{\phi}} = 0. \end{cases} \tag{90}$$

Substituting $\bar{V} = \Lambda$ in the fourth of (90) one gets $\bar{\alpha} = 0$; from the second one gets $\bar{\beta} = const$; the first and the third turn out to be identically verified. It is immediate to see that the Lagrangian (72) presents a Noether symmetry, since it does not depend on $\bar{\phi}$; being, in this particular case, $\mathcal{L}_{\bar{X}_{\bar{t}}}\bar{a} = \bar{\alpha} = 0$, this is compatible, for what we have already said, with the presence of a cyclic coordinate in the Lagrangian. Performing the conformal transformation given by (80) on the Noether vector field

$$\begin{cases} \bar{\alpha} = 0 \\ \\ \bar{\beta} = \bar{\beta}_0. \end{cases} \tag{91}$$

where $\bar{\beta}_0$ is arbitrary, we obtain

$$\begin{cases} \alpha = -a\sqrt{\dfrac{2k_0}{12k_0 - 1}}\,\bar{\beta}_0 \\ \\ \beta = \phi\sqrt{\dfrac{2k_0}{12k_0 - 1}}\,\bar{\beta}_0. \end{cases} \tag{92}$$

As it has been shown in the previous section, (92) is a Noether vector field relative to the corresponding Lagrangian in the Jordan frame, with potential given by (77) and coupling given by (78). It is easy to verify that (63) holds.

iv) There is another interesting case we would like to quote in the context of inflationary models, as last example, *i.e.*

$$V = \lambda\phi^2, \quad \lambda > 0; \quad F = k_0\phi^2, \quad k_0 < 0 \tag{93}$$

in the Jordan frame. Since the coupling is the same as the previous case (cfr. (78)), the relative conformal transformation is given by (80). To obtain the corresponding potential in the Einstein frame we have to substitute (81) in the relation

$$\bar{V}(\bar{\phi}) = \frac{\lambda}{4k_0^2\phi^2(\bar{\phi})} \tag{94}$$

that is

$$\bar{V}(\bar{\phi}) = \frac{\lambda}{4k_0^2\phi_0^2} e^{-2\sqrt{\frac{2k_0}{12k_0-1}}\,\bar{\phi}}. \tag{95}$$

We see that this case corresponds in the Einstein frame to the case of the exponential potential, for which a particular solution is the power–law inflation [30] [31]. The choice of the sign that Lucchin and Matarrese do in [30] corresponds to the choice of the sign we have done at the end of Sec. 1.

Finally, we would like to make a general remark connected with all the examples we have treated, concerning the relation between the Hubble parameter in the Einstein and in the Jordan frame. Such relation is given by

$$\begin{aligned} \bar{H} &= \frac{\dot{\bar{a}}}{\bar{a}} = \\ &= \frac{1}{(-2F)}\left(-\frac{\dot{F}}{\sqrt{-2F}} + \sqrt{-2F}\frac{\dot{a}}{a}\right) \quad = \frac{\dot{F}}{2F\sqrt{-2F}} + \frac{H}{\sqrt{-2F}} \end{aligned} \tag{96}$$

in which we have used (23) and the definition of the Hubble parameter. Relation (96) is quite useful to make some considerations on the asymptotic behavior of the Hubble parameter (see also [18]): for examples, if we require an asymptotic de Sitter–behavior in both the Einstein and Jordan frame, that is, we require $\bar{H} \overset{\bar{t}\to+\infty}{\to} \bar{C}$ and $H \overset{\bar{t}\to+\infty}{\to} C$ where $\bar{C}$ and C are constants, from (96) we obtain a differential equation for the coupling F as a function of t ($t >> 0$), given by

$$\dot{F} + 2CF - 2\bar{C}F\sqrt{-2F} = 0. \tag{97}$$

Its solution is

$$F = -\frac{C^2}{2\bar{C}^2}\left[\frac{1}{1 - F_0 e^{C\,t}} - 1\right]^2 \tag{98}$$

in which F_0 is the integration constant; this is the time–behavior that F has to assume on the solution $\phi(t)$, in order to have a de Sitter asymptotical behavior in both frames. We see from (98) that, asymptotically, we recover the standard gravity.

7 Conclusions

We have analysed the conformal equivalence between Jordan frame and Einstein frame for general coupling functions and potentials, and we have seen that any NMC theory assumes the form of the Einstein theory with a scalar field as source of the gravitational field, provided the metric undergoes the conformal transformation defined by (8) and the scalar field and the potential are transformed according to (10) We see, from these transformations, that the scalar field, although being scalar with respect to the coordinate tranformations of the space–time by definition, is not conformally scalar.

We have considered such equivalence more carefully in the cosmological case, and we have seen that the conformal transformation in this case takes the form given by (23), in which also the time–coordinate is transformed. The transformation of the time–coordinate turns out to be necessary if one requires the time–coordinate in the Einstein frame to be the universal time as well. We have seen that in case one chooses the conformal time as time–coordinate the transformation defined by (8) reduces to the form (30), which can be seen as a "coordinate transformation" on the configuration space $(a,\ \phi)$; with such a choice of time–coordinate the conformal equivalence between Jordan and Einstein frame in the cosmological case turns out to be very simple to verify.

The situation changes when ordinary matter is considered besides the scalar field: the conformal equivalence in this case is broken. We have analysed the possible descriptions of NMC theories in presence of ordinary matter and we have seen how (46) could be taken as hint in the definition of the appropriate Lagrangian, in agreement with the current discussions about the compatibility of NMC theories and the different formulations of the Equivalence Principle.

We have also seen that if a Noether symmetry is present in the "point–like " cosmological Lagrangian, this is preserved by the conformal transformation which connects Jordan and Einstein frames. This has been formally formulated in the conformal time through relation (53), since in this case the problem of the redefining the time–coordinate does not exist. We have then analysed the problem in the universal time and we have seen that the Noether symmetry is preserved under the generalized form (63), which implies as well the presence of a first integral for the corresponding Lagrangian.

We have thus analysed some aspects of the conformal equivalence between Jordan frame and Einstein frame, in particular in the cosmological case. Moreover we have generalized and improved a method of solution of cosmological NMC–models, having shown that the conformal transformation considered preserves a Noether symmetry present in the "point–like" Lagrangian, in the sense of (53). The forthcoming steps will be to investigate more deeply the implications of that: it would be interesting to classify the classes of NMC theories which are solvable by this method, and also to understand if it is possible to characterize the inflationary solutions in such context, on one side; to apply this method and to analyse the phenomenology of the models with couplings and potentials of physical interest, which can be solved, on the other side. This can be done also in connection with the problem of an opportune redefinition of the "cosmological

constant", which in this context can be time–dependent ([18] see also [32]).

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Implications of a Dilaton in Gauge Theory and Cosmology

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Abstract: After a review of theoretical motivations to consider theories with direct couplings of scalar fields to Ricci and gauge curvature terms, we consider the dynamics and non–perturbative stabilization of a dilaton in three and in four dimensions. In particular, we derive generalized Coulomb potentials in the presence of a dilaton and discuss a low energy effective dilaton potential induced by instanton effects and the S–dual coupling to axions. We conclude with a discussion of cosmological implications of a light dilaton.

1 Introduction

It is a widespread belief both in elementary particle physics and in general relativity that theoretical Ansätze for physics at the Planck scale are lacking experimental relevance today and in the foreseeable future. It is indeed a generic feature of theories unifying gravity and quantum field theory to make predictions mostly for physics at the Planck scale, and eventually also for a GUT scale a few orders of magnitude below the Planck scale. However, many seriously pursued proposals for a framework of quantum gravity, including string theory, predict scalar particles which couple both to gravity and matter fields. These scalar particles must be very weakly coupled or have to acquire large masses in order to meet experimental and cosmological constraints, or there must exist other non–perturbative mechanisms to make these scalars invisible in the low energy regime.

A particularly interesting deviation from standard Einstein–Yang–Mills theory is the prediction of a dilaton. There is no unique definition of dilatons, but a common feature of dilatons appearing in different theoretical frameworks is their direct coupling to Ricci or gauge curvature terms. A typical feature of the dilaton in string theory is its exponential coupling to Yang–Mills terms in the frame where the dilaton decouples from the curvature scalar R, and emphasis in the present paper will be on a scalar field ϕ which couples to gravity and gauge fields in four dimensions through

$$\frac{1}{\sqrt{-g}}\mathcal{L} = \frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi - \frac{1}{4}\exp(\frac{\phi}{f_\phi})F_{\mu\nu}{}^{j}F^{\mu\nu}{}_{j},$$

where the mass scale f_ϕ characterizes the strength of the coupling. Following usual conventions we denote the coupling scale f_ϕ also as a decay constant, since in many formulas it appears similar to a decay constant of a scalar bound state. However, there is an important difference: The dilatation current containing a piece $f_\phi\partial_\mu\phi$ does not parametrize a physical decay amplitude of fundamental string or Kaluza–Klein dilatons, and f_ϕ enters only with negative powers in actual transition amplitudes.

In a different setting fundamental scalars are introduced with a direct coupling to the curvature scalar to mimic a time dependent gravitational constant, to serve as an additional gravitational degree of freedom, or for the sake of local scale invariance. A scalar coupling both to the Ricci scalar and to the Maxwell term was investigated for a long time by Jordan and his collaborators. Jordan's motivation originated from Dirac's proposal of a variable gravitational constant [40], and from the observation that Kaluza–Klein theory supports Dirac's proposal if the four–dimensional metric is directly induced from a five–dimensional metric without rescaling. The coupling of the scalar to a Maxwell term is then a direct consequence of Kaluza–Klein theory. However, later Jordan abandoned the coupling to electrodynamics, thus anticipating a particular case of a Brans–Dicke theory of gravity [61].

Motivated from Mach's principle Brans and Dicke introduced a scalar $\Phi_{\mathcal{BD}}$ coupling through $R\Phi_{\mathcal{BD}}$ to account for a long range scalar field participating in the gravitational interaction [16]. In order to maintain the property that gravitational fields can be gauged away locally, they required from the outset ordinary metric couplings of all matter fields,

thus excluding in particular the coupling of the scalar to gauge fields. More general scalar–tensor theories of gravity have later been defined as theories which can be related to a Brans–Dicke theory through a $\Phi_{\mathcal{BD}}$–dependent rescaling of the metric [102], but due to the Weyl invariance of Yang–Mills terms in four dimensions these theories also contain no direct coupling to the Yang–Mills curvature.

In spite of our emphasis on couplings to Yang–Mills terms, scalar–tensor theories will be of some interest to us, due to a theoretical ambiguity in low energy string theory: It is known since long that the low energy and low curvature limit of string theory in the critical dimension is given by Einstein gravity, and this property persists in lower dimensions if low–dimensional metrics are embedded appropriately in higher–dimensional metrics. However, the claim that the leading curvature term in four–dimensional string effective actions is given by the Einstein–Hilbert term was challenged recently by Gasperini and Veneziano, see [50, 49] and references there. A compactification which rescales the low–dimensional metric by the string dilaton instead of a Kaluza–Klein dilaton gives a Brans–Dicke type theory in the curvature sector, and the problem whether low energy gravity in string theory is described by Einstein gravity in the Einstein frame or by a scalar–tensor theory in the string frame is an experimental issue.

The gravitational sector of a scalar–tensor theory with constant Brans–Dicke parameter $\omega_{\mathcal{BD}}$ reads

$$\mathcal{L}_{\mathcal{BD}} = \frac{1}{2}\sqrt{-g}\Phi_{\mathcal{BD}}\Big(R - \omega_{\mathcal{BD}} g^{\mu\nu}\partial_\mu \ln(\Phi_{\mathcal{BD}})\cdot\partial_\nu \ln(\Phi_{\mathcal{BD}}) + 2\Lambda(\Phi_{\mathcal{BD}})\Big) \tag{1}$$

$$= \sqrt{-g}\Big(\frac{1}{8|\omega_{\mathcal{BD}}|}\phi^2 R - \mathrm{sgn}(\omega_{\mathcal{BD}})\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi + \Lambda(\frac{\phi^2}{4|\omega_{\mathcal{BD}}|})\frac{\phi^2}{4|\omega_{\mathcal{BD}}|}\Big).$$

We have to be careful with the sign of $\omega_{\mathcal{BD}}$ since the Brans–Dicke parameter can attain negative values. We will see in section 3 that a "strong" version of Kaluza–Klein theory yields Brans–Dicke parameters $0 > \omega_{\mathcal{BD}} > -1$, and string theory in the string frame has $\omega_{\mathcal{BD}} = -1$.

A Brans–Dicke type coupling of a scalar field also emerged in constructions of locally Weyl invariant theories without a Weyl vector, see [105, 118, 42, 41] and references there. Coupling of a massless scalar ϕ to other particle masses through the substitution

$$m \to m\phi$$

ensures Weyl invariance with standard kinetic terms if the rescaling of the metric is accompanied by appropriate rescalings of scalar and spinor fields. Then local scale invariance can be ensured to first order if $\partial_\mu \ln\phi$ is used as a connection and if the kinetic term for ϕ corresponds to a Brans–Dicke theory in the singular limit $\omega_{\mathcal{BD}} \to -\frac{3}{2}$.

Both in scalar–tensor theories and in Weyl invariant theories the dilaton ϕ does not couple to Yang–Mills terms. However, it does couple both to Ricci and gauge curvature terms in string theory in the string frame and in Kaluza–Klein theory without metric rescaling, whereas it decouples from the Ricci scalar in Weyl transformed Kaluza–Klein theory and

in string theory in the Einstein frame:

Theory	Curvature term	Yang–Mills term	Mass terms
String theory, string frame	$\phi^2 R$	$\phi^2 F^2$	m
String theory, Einstein frame	R	$\exp(\phi/f_\phi)F^2$	m
Brans–Dicke theory	$\phi^2 R$	F^2	m
Weyl invariant theory	$\phi^2 R$	F^2	$m\phi$

Table 1: Theories with fundamental scalars coupling to curvature terms.

In the normalization of these terms a canonical kinetic term for the scalar ϕ was assumed. The "strong" and "weak" versions of Kaluza–Klein theory are related to string theory in the string frame and in the Einstein frame, respectively. In neglecting mass dependent couplings of the dilaton in the string and Kaluza–Klein framework we assumed that the mass terms originate at a scale far below any string or compactification scale. Otherwise couplings to mass terms would also appear in this sector, see section 3.

Generically different rows in this table can be connected through field dependent rescalings of the metric. This can be a useful mathematical tool in analyzing e.g. equations of motion. However, it must be stressed that field dependent rescalings of metrics are not symmetry transformations of one and the same physical system, but generically have different physical implications.

It must also be emphasized that Table 1 is by no means exhaustive: A theory may contain several dilatonic degrees of freedom, and a particularly important example is compactified heterotic string theory: This contains besides the model independent string dilaton ϕ_S at least one Kaluza–Klein dilaton ϕ_K, and a linear combination ϕ of the two couples to four–dimensional gauge fields, whereas in non–rescaled compactification or in the string frame ϕ_K or ϕ couple to R, respectively, while no dilaton couples to R in the Einstein frame. Furthermore, in recent years string theory motivated discussions of more complicated coupling functions both in the curvature and in the Yang–Mills sector.

Couplings of light scalars to Ricci or Yang–Mills curvature have very interesting cosmological implications, and in particular a direct coupling of a scalar to the Einstein–Hilbert term can have drastic consequences like removing the initial singularity of space–time without a string threshold. Cosmological implications of a Brans–Dicke scalar with and without a cosmological function Λ were discussed in [108]. String theory in the string frame has a dilaton with a Brans–Dicke type coupling, and the work of Veneziano and his collaborators attracted much interest in the resulting Brans–Dicke type cosmology known as string cosmology [50]. The cosmological implications of a dilaton in Weyl invariant theories were investigated in [78, 21]. For a disussion of a low energy string dilaton in the Einstein frame, see e.g. [34, 35]. There a duality invariant coupling between the dilaton and the axion played a crucial role in identifying a mass generating mechanism for the dilaton. However, a possible cosmological significance of axion–dilaton interactions was emphasized already in [20],

where the common appearance of axions and dilatons was motivated from supersymmetry [112, 31].

Superstring theory currently undergoes a major change of paradigm: Numerous duality symmetries have been established or proposed between seemingly different versions of the theory in various macroscopic dimensions, and it has become particularly clear that symmetry transformations interchanging axions and dilatons play a significant and fundamental role in the theory. These symmetries go by the name strong/weak coupling duality, or S–duality for short, because they involve sign flips of the dilaton. Since the expectation value of the exponentiated dilaton determines the strength of the string coupling, those sign flips can interchange strongly coupled regimes of string theory with weak coupling regimes.

We will see that under a certain constraint on decay constants of the axion and the dilaton to be explained in section 3, a low energy imprint of S–duality can still mix the dilaton and the axion. This indicates in particular that a fundamental light axion should come with a dilaton, and that their properties should be intimitely connected. Now it is known since long that the gauge theory of strong interactions [48] strongly motivates considerations of a light axion, and that instantons create an effective potential for this axion. If the QCD axion is a fundamental pseudo–scalar, and if axion–dilaton duality is realized at some scale, then we should also expect a fundamental dilaton coupling to QCD. The main motivation for the present work was the observation that instantons induce an interesting potential for such a dilaton far below the scale of supersymmetry breaking. Such a possibility was not pursued before, since an estimate on the coupling of a QCD dilaton to nucleons seems to violate the Newton approximation for weak gravitational fields, see [43] and references there:

Constraints on dilaton masses without a direct coupling to the Ricci scalar are based on the assumption that the dilaton can be treated as a local gauge coupling which shows up in nucleon masses. This might imply a coupling of the dilaton to hadronic mass terms in low energy effective theories, eventually also inducing material dependent couplings to macroscopic bodies. An estimate derived on this basis states that the dilaton should be heavier than 10^{-4} eV to be on the safe side, too heavy for a QCD dilaton with a coupling scale f_ϕ of the order of the Planck mass[1]. The reasoning implied in the derivation of this estimate was ingenious, but a weak point concerns the interpretation of the dilaton as a local gauge coupling: We will calculate the impact of a dilaton on the classical $1/r$ interaction of gauge charges in section 5 and find that the dilaton either regularizes the Coulomb potential at a distance $r_\phi \sim f_\phi^{-1}$ or implies confinement, with an interaction potential between stationary charges raising linearly with the distance. These results indicate that the impact of a dilaton in gauge theory is not adequately approximated by an effective local gauge coupling, and it seems premature to conclude that the dilaton–gluon coupling induces a dilaton–nucleon coupling proportional to the mass terms in low energy hadron theories. The problem in how far such a coupling would amount to material dependent couplings also causes some uncertainty, and lacking a reliable quantitative picture of the emergence

[1]Our conventions for Planck units are $m_{Pl} = (8\pi G)^{-1/2} = 2.4 \times 10^{18}$ GeV, $t_{Pl} = 2.7 \times 10^{-43}$ s, $l_{Pl} = 8.1 \times 10^{-35}$ m.

of the hadron spectrum in QCD both with and without a dilaton, we are currently unable to discuss constraints from the weak equivalence principle on macroscopic bodies. On the other hand, constraints from elementary particle masses in the standard model do not arise, since the dilaton originates at string or Kaluza–Klein scales far beyond the weak scale, and therefore neither the string dilaton nor a Kaluza–Klein dilaton are expected to couple to mass terms in the standard model. However, in section 8 we will also discuss implications of a dilaton with a mass $m_\phi \geq 10^{-4}$ eV. If the mechanism of axion induced dilaton stabilization outlined in section 6 is correct this mass would correspond to a decay constant $f_\phi \leq 10^{11}$ GeV far below the Planck scale, but it would still be invisible from a particle physics point of view and have interesting cosmological implications. Of course, from a stringy point of view such a low decay constant is a puzzle, since string theory generically predicts dilaton coupling scales of the order of the Planck mass.

While the results of section 5 show that the dilaton in gauge theory can not be addressed as an effective local gauge coupling, the expectation value of the dilaton still seems to imply an ambiguity in the definition of the coupling. To elucidate this consider the Lagrangian (in flat Minkowski space)

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi \cdot \partial^\mu\phi - \frac{1}{4}\exp(\frac{\phi}{f_\phi})F_{\mu\nu}{}^j F^{\mu\nu}{}_j + \overline{\psi}(i\gamma^\mu\partial_\mu + q\gamma^\mu A_\mu - m)\psi$$

describing gauge theory with a dilaton ϕ, gauge coupling q and fermions ψ. For fixed ψ the mapping

$$q \to q'$$
$$A_\mu \to A'_\mu = \frac{q}{q'}A_\mu$$
$$\phi \to \phi' = \phi - 2f_\phi \ln(\frac{q}{q'})$$

leaves the Lagrangian invariant but implies e.g. a rescaling of the 1–gluon exchange 4–fermion amplitude $\mathcal{A}_{2\to2} \to \frac{q'^2}{q^2}\mathcal{A}_{2\to2}$. This is the gauge theory version of the problem of the running dilaton or of the constancy of constants (see e.g. [39]): Motion of the expectation value of the dilaton implies a rescaling of the gluon propagator and a corresponding rescaling of the gauge coupling measured in scattering events.

On the other hand, at least the variation of the electromagnetic fine structure constant $\alpha_{em} = \frac{e^2}{4\pi}$ over cosmic time scales is strongly constrained: Improving on an analysis by Shlyakhter [89], Damour and Dyson point out that the fine structure constant about two billion years ago, when the natural Oklo fission reactor in West Africa was active, differs from todays value at most by $\frac{|\delta\alpha|}{\alpha} < 1.2 \times 10^{-7}$ [26]. On larger time scales, Varshalovich et al. give $\frac{|\delta\alpha|}{\alpha} < 1.6 \times 10^{-4}$ for the variation of the fine structure constant between ultraviolet emission of high red shift ($z \sim 3$) quasars and today [99]. In a flat universe electromagnetic waves emitted at $z = 3$ travelled for almost 88% of the lifetime of the universe (i.e. they were emitted certainly more than 9 billion years ago), and if a dilaton also couples to the photon this means that some mechanism must have stabilized the expectation value of the dilaton at an early stage in the evolution of the universe.

Usual attempts to solve the stability problem for the dilaton link the generation of a low energy dilaton potential to gaugino condensation: Due to the dilaton gaugino coupling implied by supersymmetry, a gaugino condensate provides an attractive mechanism to generate a dilaton mass. However, it must be pointed out that a gaugino condensate in the most direct and simple supersymmetric Yang–Mills dilaton theory would send the dilaton expectation value to $-\infty$ rather than stabilizing it at some finite value. This is a simple consequence of the fact that the dilaton couples to gluinos with an exponential $\exp(2\phi/f_\phi)$, i.e. with the same sign as in the dilaton gluon coupling. Therefore anomalous low energy effective terms [101, 94] or duality invariant potentials have to be employed to stabilize dilatonic degrees of freedom, see e.g. [46, 75] and references there.

The puzzle can be resolved in a different way by noting that the dilaton couples with a term $\exp(-2\phi/f_\phi)$ to the kinetic energy of the axion, whence a non–vanishing variance of the axion would provide a direct and simple way to generate a dilaton mass [34, 35]. This mechanism works with or without a dilaton gluino coupling, and the discussion in section 6 concentrates on the non–supersymmetric case.

In a string inspired four–dimensional field theory the dilaton coupling to gauge fields will generically be a superposition of the massless closed string excitation accompanying the graviton and a Kaluza–Klein dilaton defined as a logarithm of the determinant of the internal metric. In compactifications of heterotic string theory, e.g., the four–dimensional dilaton is dominated by a Kaluza–Klein component with a mixing angle of 30^0 into the string dilaton.

Like the string dilaton the Kaluza–Klein dilaton comes initially without a mass. This initial absence of a dilaton potential in the low energy sector is easy to understand: Since the low energy effective fields are zero modes of the internal derivative operators, they couple to the fluctuations of the internal dimensions, but carry no remembrance of their actual size. Therefore, we may expect dilaton stabilization in the low energy sector only if there exist non–perturbative effects which are genuinely related to the number of macroscopic dimensions, since these effects would have to disappear upon decompactification or further compactification. We know such a genuinely four–dimensional non–perturbative effect very well: Instantons in gauge theories may provide a mechanism to stabilize four dimensions, i.e. we suspect that instantons induce an effective dilaton potential in the low energy regime. However, suppose we start with some large gauge group G which decomposes into several abelian and non–abelian factors: Where do we expect the dominant instanton contributions? Instanton energies go with the inverse gauge coupling squared, but instantons are suppressed in broken gauge groups due to the large masses of the gauge fields. Therefore the dominant instanton contributions should arise in that factor of G which corresponds to the non–abelian unbroken symmetry with the largest coupling constant. If instantons stabilize the dilaton, it is QCD which has to make the dominant contribution!

Of course, there may appear other non–perturbative effects in four dimensions, eventually breaking supersymmetry and active at a much higher scale. It is well conceivable that such effects may also create an effective dilaton potential, and in a sense this is a working hypothesis for mainstream research in the field. I make no attempt to invalidate the

mainstream approach which links the dilaton potential to supersymmetry breaking, but I suppose there is enough motivation to consider an axion–dilaton system which is stabilized at or above the QCD phase transition, around 10^{-4} seconds after the big bang.

We have defined the dilaton in four dimensions through its characteristic coupling to gauge fields, yet we have to re–address its coupling to gravity, in order to explain why we exclude such couplings in the present paper: The dilaton can couple to the Einstein–Hilbert term through a term $U(\phi)R$, and we have pointed out already that direct compactification of dimensions would predict a polynomial coupling of Kaluza–Klein dilatons both to R and to gauge fields. In the gravitational sector this would correspond to a Brans–Dicke type theory of gravity with a Brans–Dicke parameter $\omega_{\mathcal{BD}} \simeq -1$ (see section 3). However, through appropriate Weyl rescalings the dilaton can be arranged to couple to gravity in a standard way without coupling directly to the curvature scalar R. The set of fields and the metric where the lowest order graviational action is given by the Einstein–Hilbert term $\int d^4x\sqrt{-g}R$ is the Einstein frame, and we will mainly use this frame. This is not just a matter of taste, but chosen on the basis of experimental constraints: Solar system measurements of light bending and time delay in gravitational fields are known to constrain deviations from standard Einstein gravity to less than 10^{-3}, and this imposes rather strong limits on scalar–tensor theories of gravity. It implies in particular that the inverse Brans–Dicke parameter measuring the direct coupling of light scalar fields to curvature is very small, implying in turn that even if the physical metric would not correspond to an Einstein frame, it would have to be very close to an Einstein frame. It was also mentioned before that the string dilaton does not couple to the Einstein–Hilbert term in the critical dimensions 10 and 26 [53, 18, 69].

There are other viable alternatives than directly relying on a Weyl transformed metric as the physical metric: If the Brans–Dicke scalar somehow acquires such a large mass that it effectively is frozen to a constant value, this would certainly comply with all contemporary tests of Einstein gravity. Furthermore, Damour and Nordtvedt pointed out that a scalar–tensor theory of gravity with a convex coupling function $U(\phi)$ also effectively reduces to Einstein gravity, even without a mass term [27].

In order to faciliate the comparison with calculations in other frames, and to clarify which models are related through Weyl rescalings, it is useful to have a dictionary of the behavior of various sectors in the Lagrangian under Weyl transformations: Under rescalings of the metric in D dimensions

$$\tilde{g}_{\mu\nu} = \exp\Big(2\frac{\lambda - \tilde{\lambda}}{D-2}\phi\Big) g_{\mu\nu}$$

the gravitational sector transforms according to

$$\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\exp(\tilde{\lambda}\phi)(\tilde{R}_{\mu\nu} + \frac{D-1}{D-2}\tilde{\lambda}^2\partial_\mu\phi\cdot\partial_\nu\phi) = \sqrt{-g}g^{\mu\nu}\exp(\lambda\phi)(R_{\mu\nu} + \frac{D-1}{D-2}\lambda^2\partial_\mu\phi\cdot\partial_\nu\phi).$$

In the matter sector one finds for Yang–Mills fields

$$\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\tilde{g}^{\rho\sigma}\exp(\tilde{\alpha}\phi)\mathrm{tr}(F_{\mu\rho}\cdot F_{\nu\sigma}) = \sqrt{-g}g^{\mu\nu}g^{\rho\sigma}\exp(\alpha\phi)\mathrm{tr}(F_{\mu\rho}\cdot F_{\nu\sigma})$$

with

$$\alpha = \tilde{\alpha} + \frac{D-4}{D-2}(\lambda - \tilde{\lambda}),$$

while fermions coupling to Yang–Mills fields A and a canonical spin connection Ω transform according to

$$\sqrt{-\tilde{g}}\bar{\tilde{\psi}}[\tilde{e}^{\mu}{}_{a}\gamma^{a}(\partial_{\mu}+\tilde{\Omega}_{\mu}-iqA_{\mu})+im]\tilde{\psi} = \sqrt{-g}\bar{\psi}[e^{\mu}{}_{a}\gamma^{a}(\partial_{\mu}+\Omega_{\mu}-iqA_{\mu})+\exp\Big(\frac{\lambda-\tilde{\lambda}}{D-2}\phi\Big)im]\psi$$

with

$$\tilde{\psi} = \exp\Big(-\frac{D-1}{D-2}\frac{\lambda-\tilde{\lambda}}{2}\phi\Big)\psi,$$

$$\tilde{\Omega}^{a}{}_{b\mu} = \Omega^{a}{}_{b\mu} + \frac{\lambda-\tilde{\lambda}}{D-2}(e_{\mu}{}^{a}e^{\nu}{}_{b} - e_{\mu b}e^{\nu a})\partial_{\nu}\phi.$$

As an application, the combined Weyl transformation and redefinition

$$g_{\mu\nu} = F(\Phi)^{\frac{2}{D-2}}\tilde{g}_{\mu\nu}$$

$$\phi = \int d\Phi \sqrt{\frac{G(\Phi)}{F(\Phi)} + \frac{D-1}{(D-2)\kappa}\frac{F'(\Phi)^2}{F(\Phi)^2}}$$

transform Jordan–Brans–Dicke type theories into the (physically inequivalent) Einstein frame:

$$\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}(\frac{1}{2\kappa}F(\Phi)\tilde{R}_{\mu\nu} - \frac{1}{2}G(\Phi)\partial_{\mu}\Phi\cdot\partial_{\nu}\Phi) = \sqrt{-g}g^{\mu\nu}(\frac{1}{2\kappa}R_{\mu\nu} - \frac{1}{2}\partial_{\mu}\phi\cdot\partial_{\nu}\phi),$$

and according to the previous paragraph we may also transform the kinetic terms of the fermions to standard form.

Our analysis in four dimensions will be mainly based on gauge theory coupled to an axion a and a dilaton ϕ in the Einstein frame:

$$\frac{1}{\sqrt{-g}}\mathcal{L} = \frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\cdot\partial_{\nu}\phi - \frac{1}{2}\exp(-2\frac{\phi}{f_{\phi}})g^{\mu\nu}\partial_{\mu}a\cdot\partial_{\nu}a \qquad (2)$$

$$-\frac{1}{4}\exp(\frac{\phi}{f_{\phi}})F_{\mu\nu}{}^{j}F^{\mu\nu}{}_{j} + \frac{q^2}{64\pi^2 f_a}\epsilon^{\mu\nu\rho\sigma}aF_{\mu\nu}{}^{j}F_{\rho\sigma j}.$$

Superficially, we will refer to this system as an axion–dilaton–gluon system, irrespective of whether the gauge group is SU(3) or some other Lie group.

The particular ratio between the couplings of the dilaton to gauge fields and the axion in (2) can be motivated for two reasons: On the one hand, this ratio of couplings arises automatically in Kaluza–Klein compactifications to four dimensions, as will be elucidated in section 3, while on the other hand the same ratio also arises from the requirement of duality symmetry in the axion–dilaton system. The last, and maybe most important motivation for consideration of the axion–dilaton–gluon system comes from the observation

that this system must hold a clue to the issue of dilaton stabilization in four dimensions: It is known that instantons create an effective axion potential $V(a) \sim -\cos(a/f)$ plus higher order terms in $\cos(a/f)$. We will see that instantons and the effective axion potential survive introduction of the dilaton and break the scale invariance of the equations of motion following from (2). This provides a strong hint that instantons and axions must also lift the degeneracy of the dilaton potential associated with the scale invariance of the tree level equations of motion.

The resulting dilaton potential has many interesting implications on the low energy sector of string theory and cosmology. In particular, the dynamics of the dilaton switches from expansion dominance to an oscillatory behavior around 10^{-4} seconds after the initial singularity, that is "long" after the onset of axion oscillations ($\sim 10^{-6}$ seconds). At this time the temperature of the universe has already dropped to a value near the QCD phase transition, and the dilaton can make an appreciable contribution to the energy density of the universe as a cold dark matter candidate if its variance above 1 TeV is of the order of the Planck mass $\sqrt{\phi^2} \sim m_{Pl}$.

Investigation of the dilaton potential resulting from the observations outlined above and discussions of cosmological implications will be a primary concern in the present work, and we will concentrate on these tasks especially in sections 6–8. Besides this, we will discuss stabilization of the dilaton in three dimensions in section 4, while the impact of the dilaton on potentials of pointlike particles in four–dimensional gauge theory is examined in section 5. Sections 2 and 3 survey introductory material needed in the discussion of the dilaton potential. While no attempt was made to make the paper fully self–contained, the introductory sections should make it amenable to non–experts with some basic knowledge in quantum field theory, string theory and cosmology.

2 Instantons in gauge theory

Instantons are classical solutions of Yang–Mills equations on Euclidean four–dimensional manifolds [12, 59, 3]. They play a prominent role in particle physics through their contribution to 't Hooft's solution of the strong U(1) problem and through their contribution to chiral symmetry breaking: Chiral symmetry breaking is signaled through a quark condensate which can be related to the spectral density $\rho(\lambda)$ of the Euclidean Dirac operator through a celebrated relation of Banks and Casher [7]

$$\langle \overline{q} q \rangle = -\pi \rho(0).$$

While we are still lacking a full understanding of the dynamics of chiral symmetry breaking in QCD from first principles, the emergence of a non–vanishing spectral density at eigenvalue zero can be attributed to the presence of an instanton liquid in the Euclidean vacuum, as has been thoroughly reviewed in [82, 91].

In the present setting we are interested in instantons because some appropriate generalization of them will contribute to the effective potential of the dilaton, and we will review some of the properties of instantons in this section. Classical and very thorough reviews of instantons have been given in [23, 98]. For the ADHM construction of the general multi–instanton solution see [3] and [24] and references there. The path breaking work on the calculation of quantum effects was [59].

In this introductory section we will only review the one–instanton solution elaborating on a theorem of Wilczek. Wilczek's theorem relates instantons to conformally flat spaces of constant scalar curvature [109, 19, 32] and has the virtue to naturally yield 't Hooft's Ansatz and to explain the group theoretic origin of the 't Hooft symbols [59].

Throughout this section we will work with two conformally related Euclidean 4–spaces, one of those being flat while the other metric can be written as

$$g_{\mu\nu}(x) = \chi^2(x)\delta_{\mu\nu} = \frac{1}{\sigma^2(x)}\delta_{\mu\nu}.$$

Indices raised with $g^{\mu\nu}(x)$ will be denoted by a dot in this section: $g^{\mu\nu}(x)v_\nu(x) = v^{\dot{\mu}}(x)$. The gauge covariant derivative is[2] $D_\mu = \partial_\mu + \mathcal{A}_\mu$ and the covariant derivative in the conformally flat space is ∇_μ. Since we will not perform Legendre transformations in the Euclidean setting and the Euclidean partition function resembles a canonical ensemble, we will use the terms action and energy synonymously in this section.

The well-known local isomorphism $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ may be used to reduce $\mathrm{gl}(4,\mathbb{R})$ connections Γ to su(2) connections ${}^{\pm}\mathcal{A}$ according to

$${}^{\pm}\mathcal{A}_{\mu i} = -({}^{\pm}Z_i)^{\alpha}{}_{\beta}\Gamma^{\beta}{}_{\alpha\mu}, \tag{3}$$

[2] We are using an anti–hermitian basis for $\mathcal{A}$. The relation between the gauge potential $\mathcal{A}_\mu \equiv \mathcal{A}_{\mu i}(-iX^i)$ and $A_\mu \equiv A_{\mu i}X^i$ is $\mathcal{A}_{\mu i} = qA_{\mu i}$.

where an explicit representation of the SU(2) projectors is

$$({}^{\pm}Z_i)_{\mu\nu} = \frac{1}{2}(\pm\delta^0{}_\nu\delta_{i\mu} \mp \delta^0{}_\mu\delta_{i\nu} - \epsilon_{ijk}\delta^j{}_\mu\delta^k{}_\nu).$$

These projectors provide a selfdual and an anti–selfdual basis for four–dimensional representations of su(2), and in compliance with the uniqueness of the spin–$\frac{3}{2}$ equivalence class, they are intertwined via $\mathbb{T}\cdot{}^{+}Z_i\cdot\mathbb{T} = {}^{-}Z_i$ with $\mathbb{T} = \mathrm{diag}\{-1,1,1,1\}$. Identifying indices which carry the values 0 or 4, these generators are related to the 't Hooft symbols via

$$({}^{+}Z_i)_{\mu\nu} = -\frac{1}{2}\overline{\eta}_{i\mu\nu}, \qquad ({}^{-}Z_i)_{\mu\nu} = -\frac{1}{2}\eta_{i\mu\nu}.$$

Thus the appendix of [59] may be used if some signs are adjusted properly due to $\epsilon_{0123} = -\epsilon_{1234}$. The reduction according to (3) yields su(2)–valued curvatures

$${}^{\pm}\mathcal{F}_{\mu\nu}{}^{i} = -({}^{\pm}Z^i)^\alpha{}_\beta(R^\beta{}_{\alpha\mu\nu} + S^\beta{}_{\alpha\mu\nu}) \tag{4}$$

with the deviation from the Riemannian

$$S^\alpha{}_{\beta\mu\nu} = \frac{1}{2}\Gamma^\rho{}_{\beta\mu}(\Gamma^\alpha{}_{\rho\nu} + \Gamma_\rho{}^\alpha{}_\nu) - \frac{1}{2}\Gamma^\alpha{}_{\rho\mu}(\Gamma^\rho{}_{\beta\nu} + \Gamma_\beta{}^\rho{}_\nu).$$

The conformal Ansatz is now introduced by inserting the Levi–Civitá connection

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2\sigma^2}(\delta_{\mu\nu}\partial^\rho\sigma^2 - \delta^\rho{}_\mu\partial_\nu\sigma^2 - \delta^\rho{}_\nu\partial_\mu\sigma^2) \tag{5}$$

corresponding to the conformally flat metric g. This Ansatz leads to $S = 0$, and (3) reduces to the 't Hooft Ansatz:

$${}^{\pm}\mathcal{A}_{\mu i} = ({}^{\pm}Z_i)^\rho{}_\mu\partial_\rho\ln(\sigma^2). \tag{6}$$

The combination of ${}^{\pm}\mathcal{A}_\mu{}^i$ and Γ into a generally covariant derivative:

$$\mathcal{D}_\mu{}^{\pm}\mathcal{F}_{\lambda\nu} = \partial_\mu{}^{\pm}\mathcal{F}_{\lambda\nu} + [{}^{\pm}\mathcal{A}_\mu, {}^{\pm}\mathcal{F}_{\lambda\nu}] - {}^{\pm}\mathcal{F}_{\lambda\sigma}\Gamma^\sigma{}_{\nu\mu} - {}^{\pm}\mathcal{F}_{\sigma\nu}\Gamma^\sigma{}_{\lambda\mu} \tag{7}$$

leaves the su(2)–projectors invariant:

$$\mathcal{D}_\mu({}^{\pm}Z_i)_{\lambda\nu} = \nabla_\mu({}^{\pm}Z_i)_{\lambda\nu} + \epsilon_{ijk}{}^{\pm}\mathcal{A}_\mu{}^j({}^{\pm}Z^k)_{\lambda\nu} = 0.$$

As a consequence covariant differentiation and su(2) reduction commute:

$$\mathcal{D}_\mu{}^{\pm}\mathcal{F}_{\lambda\nu} = -({}^{\pm}Z_i)^\alpha{}_\beta\nabla_\mu R^\beta{}_{\alpha\lambda\nu}. \tag{8}$$

Furthermore, the gauge covariant divergence and the generally covariant divergence of the field strength are related in a simple way:

$$D_\mu{}^{\pm}\mathcal{F}^\mu{}_{\nu i} = \frac{1}{\sigma^2}\mathcal{D}_\mu{}^{\pm}\mathcal{F}^{\dot\mu}{}_{\nu i}. \tag{9}$$

Equations (8,9) imply a simple relation between the divergences of $\mathcal{F}$ and R:

$$D_\mu{}^{\pm}\mathcal{F}^{\mu}{}_{\nu i} = -\frac{1}{\sigma^2}({}^{\pm}Z_i)^{\alpha}{}_{\beta}\nabla^{\dot{\mu}}R^{\beta}{}_{\alpha\mu\nu} \tag{10}$$

and this yields a translation of the Yang–Mills equations into a condition on the curvature of the conformally flat space. The vanishing of the Weyl tensor implies that the covariant divergence of the Riemannian in a conformally flat space is

$$\nabla^{\dot{\mu}}R^{\beta}{}_{\alpha\mu\nu} = -\frac{1}{6}(\delta^{\beta}{}_{\nu}\partial_\alpha - \delta_{\alpha\nu}\partial^\beta)R$$

with the curvature scalar $R = R^{\dot{\mu}}{}_{\mu}$. Therefore the gauge covariant divergence of the su(2) curvatures from the conformally flat space are simply proportional to the gradient of the curvature scalar:

$$D_\mu{}^{\pm}\mathcal{F}^{\mu}{}_{\nu i}(x) = \frac{1}{3\sigma^2(x)}({}^{\pm}Z_i)^{\mu}{}_{\nu}\partial_\mu R(x) \tag{11}$$

and this expresses Wilczek's theorem [109]: su(2) reduction of the Riemannian connection in a conformally flat 4–space yields solutions to the Yang–Mills equations if and only if the curvature scalar is constant.

If the curvature scalar is expressed in terms of the conformal factor χ this theorem establishes a connection between Yang–Mills theory and χ^4 theory:

$$\partial_\mu\partial^\mu\chi + \frac{1}{6}R\chi^3 = 0. \tag{12}$$

However, the formulation in terms of the inverse conformal factor serves our purposes better:

$$\partial_\mu\partial^\mu\sigma^2 - \frac{3}{2\sigma^2}\partial_\mu\sigma^2\cdot\partial^\mu\sigma^2 - \frac{1}{3}R = 0. \tag{13}$$

We can not solve this equation in a general fashion. However, it was observed in [32] that the stronger condition of local symmetry

$$\nabla_\lambda R^{\dot{\mu}}{}_{\nu} = 0$$

allows for a general solution. In terms of $\sigma(x)$ the condition of local symmetry takes the following form:

$$\partial^\mu\partial_\nu\partial_\lambda\sigma^2 + \delta^{\mu}{}_{\nu}\partial_\lambda[\sigma\partial_\alpha\partial^\alpha\sigma - 2(\partial_\alpha\sigma)(\partial^\alpha\sigma)] - [\delta^{\mu}{}_{\nu}\partial_\lambda + \delta^{\mu}{}_{\lambda}\partial_\nu + \delta_{\lambda\nu}\partial^\mu](\partial_\alpha\sigma)(\partial^\alpha\sigma) = 0. \tag{14}$$

Summation over μ and ν leads again to

$$\partial_\lambda R = 6\partial_\lambda[\sigma\partial_\alpha\partial^\alpha\sigma - 2(\partial_\alpha\sigma)(\partial^\alpha\sigma)] = 3\partial_\lambda[\partial_\alpha\partial^\alpha\sigma^2 - 6(\partial_\alpha\sigma)(\partial^\alpha\sigma)] = 0 \tag{15}$$

and this shows that equation (14) is equivalent to a set of equations consisting of (13) and

$$\partial_\mu\partial_\nu\partial_\lambda\sigma^2 - \frac{1}{6}(\delta_{\mu\nu}\partial_\lambda + \delta_{\mu\lambda}\partial_\nu + \delta_{\lambda\nu}\partial_\mu)\partial_\alpha\partial^\alpha\sigma^2 = 0. \tag{16}$$

Eq. (16) can be solved by standard methods, and the simplest method of solution assuming differentiability to fifth order proceeds as follows:

Contracting (16) with ∂^λ and with $\partial^\lambda\partial^\nu$ yields

$$\partial_\mu\partial_\nu\partial_\lambda\partial^\lambda\sigma^2 = \frac{1}{4}\delta_{\mu\nu}(\partial_\lambda\partial^\lambda)^2\sigma^2$$

$$\partial_\mu(\partial_\lambda\partial^\lambda)^2\sigma^2 = 0$$

implying

$$\partial_\mu\partial_\nu\partial_\rho\partial^\rho\sigma^2 = 48\lambda^2\delta_{\mu\nu}$$

where in writing the arbitrary integration constant as a square we already took into account that our solution in the end should correspond to a definite function. The previous equation is readily integrated to

$$\partial_\nu\partial_\lambda\partial^\lambda\sigma^2 = 48\lambda^2 x_\nu + 12c_\nu$$

whence (16) reduces to

$$\partial_\mu\partial_\nu\partial_\rho\sigma^2 = \delta_{\mu\nu}(8\lambda^2 x_\rho + 2c_\rho) + \delta_{\mu\rho}(8\lambda^2 x_\nu + 2c_\nu) + \delta_{\rho\nu}(8\lambda^2 x_\mu + 2c_\mu).$$

This is readily integrated again and the general solution contains 20 parameters:

$$\sigma^2(x) = \lambda^2 r^4 + c_\mu x^\mu r^2 + \frac{1}{2}a_{\mu\nu}x^\mu x^\nu + b_\mu x^\mu + \zeta^2. \tag{17}$$

We may identify the 4–vector c and the off–diagonal components of a as Poincaré degrees of freedom, and gauge them away through an appropriate Poincaré transformation. We then write $a_{\mu\nu} = \Lambda_\mu\delta_{\mu\nu}$ and use the abbreviation $A \equiv \Sigma_\mu\Lambda_\mu - \frac{R}{3}$. In this gauge equation (13) translates into the following set of algebraic constraints on the coefficients in (17):

$$\begin{aligned} A\lambda^2 &= 0, \\ 2A\zeta^2 &= 3b_\mu b^\mu, \\ \lambda^2 b_\mu &= 0, \\ (3\Lambda_\mu - A)b_\mu &= 0, \\ (3\Lambda_\mu - A)\Lambda_\mu &= 48\lambda^2\zeta^2. \end{aligned}$$

In case $\lambda^2 = 0$ there exist only singular solutions with either a pointlike singularity (corresponding to the meron solution of De Alfaro, Fubini and Furlan [29]), singular lines, planes or 3–spaces. In case $\lambda^2 > 0$ there exist singular solutions with either two pointlike singularities (2–meron solution), a singular circle, a singular 2–sphere, or a singular 3–sphere. However, there exists one regular solution given by

$$\sigma(x) = \lambda(r^2 + \varrho^2). \tag{18}$$

The corresponding metric describes a 4–sphere of radius $r_4 = \frac{1}{2\lambda\varrho}$ centered at a point $(\varrho - r_4)\vec{e}_5$ in $\mathbb{R}^5$ in stereographic coordinates. The BPST instanton ${}^{+}\mathcal{A}$ and anti–instanton ${}^{-}\mathcal{A}$ are

$$ {}^{\pm}\mathcal{A}_{\mu i} = \frac{2}{r^2+\varrho^2}(\pm\delta^0{}_\mu x_i \mp \delta_{i\mu}x^0 + \epsilon_{ijk}\delta^j{}_\mu x^k) \tag{19} $$

and they satisfy

$$ \lim_{r\to\infty} {}^{+}\mathcal{A}_{\mu i}(x)\frac{\sigma^i}{2i} = U^{-1}(x)\partial_\mu U(x), \qquad \lim_{r\to\infty} {}^{-}\mathcal{A}_{\mu i}(x)\frac{\sigma^i}{2i} = U(x)\partial_\mu U^{-1}(x), $$

with

$$ U(x) = \frac{1}{r}(x^0 + x^i\sigma_i). $$

Writing the angle to $\vec{e}_0$, $\cos\vartheta = \frac{x^0}{r}$, it is apparent that

$$ U^{\pm n}(x) = \cos(n\vartheta) \pm i\frac{x^i\sigma_i}{\sqrt{x_j x^j}}\sin(n\vartheta) $$

describe mappings of winding numbers $\pm n$ from S^3 to SU(2).

The corresponding field strengths are

$$ {}^{\pm}E_j{}^i = \pm\frac{4\varrho^2}{q(r^2+\varrho^2)^2}\delta_j{}^i, \qquad {}^{\pm}B_j{}^i = -\frac{4\varrho^2}{q(r^2+\varrho^2)^2}\delta_j{}^i, \tag{20} $$

and these solutions are apparently (anti–)selfdual under the euclidean duality transformation $E \to -B$, $B \to -E$. In the present construction the duality property arises as a consequence of the diagonal structure of the field strengths with respect to space–time and su(2) indices. This ensures that the duality of ${}^{\pm}\mathcal{F}^\alpha{}_{\beta\mu\nu} \equiv {}^{\pm}\mathcal{F}^i{}_{\mu\nu}({}^{\pm}Z_i)^\alpha{}_\beta$ in the internal indices α, β carries over to the space–time indices μ, ν.

The energy density is

$$ \mathcal{L} = \frac{48\varrho^4}{q^2(r^2+\varrho^2)^4} $$

yielding an action

$$ S = \frac{8\pi^2}{q^2}. $$

These solutions and their generalizations to higher winding numbers n appear also in general SU(N_c) gauge theory through the various embeddings of SU(2) subgroups.

It is apparent then that instantons should contribute to the effective dilaton potential since the dilaton couples directly to F^2 and a large dilaton would imply large instanton energy.

The singular solutions with $\lambda^2 \neq 0$ also fall off like r^{-4} at large distances and therefore really correspond to solutions of the vacuum Yang–Mills equations, albeit with infinite action.

3 The Kaluza–Klein paradigm

Kaluza–Klein theory asserts that part of the scalar and vector fields and the metric in a theory in d dimensions can be identified as components of a higher–dimensional metric, and that the appropriate Lagrangian in d dimensions can be inferred from a higher–dimensional theory containing additional compact dimensions. One might distinguish a strong and a weak version of Kaluza–Klein theory: The strong version would suppose that the d–dimensional inverse metric tensor is directly embedded in the corresponding higher–dimensional inverse metric without rescaling. This would require a direct coupling of scalars to the Einstein–Hilbert term in the low–dimensional theory. The weak version would permit field dependent reparametrizations between the actual d–dimensional metric and the metric inherited from higher dimensions.

In the more recent history of physics, interest in the dilaton revived in the realm of Kaluza–Klein supergravity. If the field content of space–time is assumed to arise from embeddings in a $(4+d_i)$–dimensional manifold of Minkowski signature with a compact factor, the d_i–dimensional internal manifold will have variable volume from the four–dimensional point of view, and this variable volume will be encoded in a local field $\phi(x)$, the dilaton. However, my expectation is that a low–dimensional Kaluza–Klein dilaton tells us only about local variations of the volume of the internal manifold, its four–dimensional dynamics can not fix the mean value of that volume. Explaining and fixing the smallness of internal dimensions belongs to the realm of Kaluza–Klein cosmology or string theory in the $(4+d_i)$–dimensional framework. Nevertheless, the low–dimensional field theory must be self–consistent and tell us why variations of the internal dimensions are small or invisible, and this problem concerns the issue of the effective potential of Kaluza–Klein type dilatons in four dimensions.

While Kaluza–Klein theory is not a suitable framework for a unified theory of fundamental interactions, it still provides an indispensable tool in field theoretical investigations of string theory, where compact manifolds with or without boundaries arise both in low energy effective field theories and in the net of extended objects with p spatial dimensions, so called p–branes. The dynamics of nets of various p–branes in $D \leq 11$ dimensions is currently under close investigation as a paradigm for non–perturbative effects in string theory. Far away from the intersections the excitations of the p–branes can be described in terms of $p+1$–dimensional field theories on the branes, and compactness arises for those dimensions which describe the extension of a given p–brane between other extended objects.

Quantum field theory on manifolds with compact directions is an old subject of theoretical physics, with an almost canonical structure in the bosonic sector and some matters of taste in the more complicated fermionic sector. I will give an account on the compactification of one dimension in my favorite conventions, and the cogniscenti may find it amusing (or bothersome) to compare to their favorite frameworks. We will go from D to $D-1$ dimensions through compactification on a circle, and assume $D \geq 4$ in the sequel. As long as only zero modes of internal derivatives are taken into account repeated application of the following formulas can be used to parametrize any Kaluza–Klein theory ending up in $d \geq 3$

dimensions. However, besides compactifications from 5 to 4 and from 4 to 3 dimensions we will need only a few selected results for compactifications from $D \geq 6$ to 4 dimensions.

In addition to serving as a generator for torus compactifications there are further reasons to go from D to $D-1$ dimensions: My first motivation for this arose from recent developments in string theory: Motivated by the succesful applications of duality symmetries in the investigation of low energy effective actions of supersymmetric gauge theories, Witten observed that theories with Bogomol'nyi saturated solitons should have dual descriptions in terms of Kaluza–Klein theories, in order to explain the equidistant mass spectrum of particles in the dual theory [113, 114]. This fits very well with proposals by Witten and by Hořava and Witten, saying that large dilaton limits of type *IIA* and heterotic $E_8 \times E_8$ superstring theory are described by large radius compactifications of an eleven–dimensional theory on $M_{10} \times S^1$ or $M_{10} \times S^1/\mathbb{Z}_2$, which has been called M–theory [113, 60]. While this initiated the current activity to define or identify M–theory as a matrix theory or as a theory treating extended objects of various dimensions as (almost) equally fundamental degrees of freedom[3], it also motivated speculations about relations between supersymmetric theories in three dimensions and non–supersymmetric theories in four dimensions, including in particular a proposal to solve the cosmological constant problem through supersymmetry in three dimensions [115]. While there is no complete picture yet concerning the impact of different sectors of the moduli space of M–theory and string theory on low energy physics, it implies existence of parametrizations of low energy effective field theory which implement a dilaton either in three or in four dimensions[4]. It has been stressed by Banks and Dine that four–dimensional compactifications of M–theory should imply a five–dimensional threshold two orders of magnitude below the GUT scale, i.e. around 10^{14} GeV [8], implying that quantum gravity effects may become strong before unification is achieved in a field theoretic framework. In a recent update on Kaplunovsky's work on large volume string compactifications Caceres et al. also discuss a five–dimensional threshold in heterotic string theory [22].

A Kaluza–Klein decomposition of the metric which preserves the Einstein–frame is

$$G_{MN} = \Phi^{-\frac{1}{D-2}} \begin{pmatrix} g_{\mu\nu} + \Phi a_\mu a_\nu & \Phi a_\mu \\ \Phi a_\nu & \Phi \end{pmatrix}. \tag{21}$$

A priori this is a mere reparametrization of the metric comparable to an ADM decomposition. It becomes an Ansatz if we assume that the fields do not depend on x^{D-1} or their dependence on x^{D-1} is negligible, due to translational invariance or compactness along

[3] My impression is that strings still play a more fundamental role than other extended objects, since their spectra determine the brane–scan.

[4] Even in the Einstein frame a radially coupled dilaton in a field theory on $M_d \times X \times S^1$ looks like a radially coupled dilaton from an Einstein frame on $M_d \times S^1$ only in a very particular class of parametrizations of the Kaluza–Klein Ansatz. Generically, the large dilaton limit will look like a scalar–vector–tensor theory of gravity on $M_d \times S^1$.

x^{D-1}. Eq. (21) then yields for the zero modes of ∂_{D-1} and up to surface terms[5]

$$\sqrt{-G}\, G^{MN} R_{MN} = \sqrt{-g}\, [g^{\mu\nu} R_{\mu\nu} - \frac{D-3}{4(D-2)} g^{\mu\nu} \partial_\mu \ln \Phi \cdot \partial_\nu \ln \Phi - \frac{\Phi}{4} f_{\mu\nu} f^{\mu\nu}], \qquad (22)$$

while reduction of the Yang–Mills field $A_M{}^k \to A_\mu{}^k$, A^k yields

$$-\frac{1}{4}\sqrt{-G}\, F_{MN}{}^k F^{MN}{}_k = -\frac{1}{4}\sqrt{-g}\, \Phi^{\frac{1}{D-2}} [B_{\mu\nu}{}^k B^{\mu\nu}{}_k + \frac{2}{\Phi} g^{\mu\nu} D_\mu A^k \cdot D_\nu A_k], \qquad (23)$$

$$B_{\mu\nu} = F_{\mu\nu} + a_\mu D_\nu A - a_\nu D_\mu A.$$

We follow the usual terminology to denote the mechanism leading from Einstein gravity in D dimensions to Einstein–Yang–Mills theory in $D-1$ dimensions as compactification on a circle. This has to be qualified in three directions: First one should keep in mind that Einstein gravity actually induces a scalar–vector–tensor type theory of gravity on submanifolds, which is not strictly a scalar–vector–tensor theory since the scalar and vector degrees of freedom also couple directly to matter. In the scalar–tensor sector the induced theory would resemble a Jordan–Brans–Dicke theory as a consequence of the fact that the metric on the submanifold induced from the metric G is

$$\tilde{g}_{\mu\nu} = \Phi^{-\frac{1}{D-2}} (g_{\mu\nu} + \Phi a_\mu a_\nu).$$

If we insist that Kaluza–Klein theory yields Einstein gravity in lower dimensions we suppose that the physical metric on the submanifold is not the metric inherited from the embedding space. Stated differently, the geodesics traced out by test particles are not the geodesics of the embedded submanifold.

The second remark concerns the naive picture of a product manifold $\mathcal{M} \times \mathcal{X}$, with $\mathcal{X}$ compact. While this is a particular possibility considered by Kaluza–Klein theory, it is not the most general setting. Generically, the compact manifold $\mathcal{X}$ acts as a fiber in a total space which projects to $\mathcal{M}$. Indeed, in the case of a product manifold the vector fields a could be safely neglected if the Riemannian structure respects the product structure: The connection coefficients $\Gamma^{D-1}{}_{\mu\nu}$ and $\Gamma^{\mu}{}_{D-1\nu}$ are gauge equivalent to zero for arbitrary Φ if and only if a is a gradient. This is equivalent to the requirement that vectors tangent and normal to the internal dimensions are mapped to tangent and normal vectors under parallel translation.

The third remark also concerns the graviphotons a: To lowest order in scalar contributions to the metric these vector fields contribute Maxwell terms to the low–dimensional Lagrangian. However, fermions will be neutral with respect to these gauge fields. Instead, the graviphoton mixes with gauge fields inherited from the higher–dimensional theory in such a way that the low–dimensional gauge field becomes neutral under diffeomorphisms

[5] Whenever equations are written down for terms appearing in a Lagrangian, we will consider expressions equivalent if they differ by a divergence and neglect the divergences in the sequel.

Compactification to the Einstein frame becomes singular for $D = 3$, and other Ansätze have to be employed in this case, see e.g. [17, 71, 72].

normal to the submanifold. This can be seen explicitly in the reduction of fermion contributions to the Lagrangian carried out below.

A detailed discussion of the reduction of fermion terms requires a distincton between even and odd values of D. My favorite choice for embeddings and reductions of γ–matrices is based on Weyl bases in even dimensions and Dirac bases in odd dimensions. General discussions of properties of spinors in arbitrary dimensions can be found in [73, 107].

If D is **even** and γ_μ is a basis of Dirac matrices in $D-1$ dimensions, a Weyl basis of Dirac matrices in D dimensions is given by

$$\Gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Gamma_j = \begin{pmatrix} 0 & -\gamma_0\gamma_j \\ \gamma_0\gamma_j & 0 \end{pmatrix}$$

$$\Gamma_{D-1} = \begin{pmatrix} 0 & -\gamma_0 \\ \gamma_0 & 0 \end{pmatrix}.$$

The corresponding analog of γ_5 is

$$\Gamma_{D+1} = i^{\frac{D}{2}+1}\Gamma_0 \cdot \Gamma_1 \ldots \Gamma_{D-1}.$$

If we start with a $(1+0)$–dimensional γ–matrix $\gamma_0 = -1$, and with the conventions for γ–matrices in odd dimensions described below, Γ_{D+1} takes the form

$$\Gamma_{D+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This embedding of γ–matrices provides a direct mapping between the spinor representation in D dimensions and both inequivalent representations in $D-1$ dimensions, and the appropriate Ansatz for the dimensional reduction of the spinor field is

$$\Psi = \Phi^{\frac{1}{4(D-2)}} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$

In a gauge $E^\mu{}_{D-1} = 0$ for the D–*bein* the dimensionally reduced action reads

$$\sqrt{-G}\,\overline{\Psi}[E^M{}_A\Gamma^A(i\partial_M + i\Omega_M + qA_M) - M]\Psi = \tag{24}$$

$$= \sqrt{-g}\,[\overline{\psi}_+ e^\mu{}_a\gamma_+^a(i\partial_\mu + i\omega_{+\mu} + qV_\mu)\psi_+ + \overline{\psi}_- e^\mu{}_a\gamma_-^a(i\partial_\mu + i\omega_{-\mu} + qV_\mu)\psi_-]$$

$$+q\sqrt{-g}\,\Phi^{-\frac{1}{2}}(\overline{\psi}_+ A\psi_+ + \overline{\psi}_- A\psi_-) + M\sqrt{-g}\,\Phi^{-\frac{1}{2(D-2)}}(\psi_+^+\psi_- + \psi_-^+\psi_+)$$

$$-\frac{i}{8}\sqrt{-g}\,\Phi^{\frac{1}{2}} f_{ab}(\overline{\psi}_+\gamma_+^{ab}\psi_+ + \overline{\psi}_-\gamma_-^{ab}\psi_-)$$

with $\gamma_\pm^0 = \pm\gamma^0$, $\gamma_\pm^j = \gamma^j$, while

$$\Omega_\mu = -\frac{1}{4}\Gamma^A\Gamma^B\Omega_{AB\mu}$$

and

$$\omega_{\pm\mu} = -\frac{1}{4}\gamma^a_\pm \gamma^b_\pm \omega_{ab\mu}$$

denote the canonical spin connections for the metrics G and g, respectively. The vector field V_μ appearing as a gauge potential in $D-1$ dimensions is

$$V_\mu = A_\mu - a_\mu A,$$

$$V_{\mu\nu} = B_{\mu\nu} - A f_{\mu\nu}.$$

Note that $D_\mu A_j = \partial_\mu A_j - q A_\mu{}^i f_{ij}{}^k A_k = \partial_\mu A_j - q V_\mu{}^i f_{ij}{}^k A_k$.

If D is **odd** and γ_μ is a basis of Dirac matrices in $D-1$ dimensions, a basis of Dirac matrices in D dimensions is given by

$$\Gamma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Gamma_j = \gamma_j, \qquad 1 \le j \le D-2$$

$$\Gamma_{D-1} = -i\gamma_0.$$

We have

$$i^{\frac{D+1}{2}+1}\Gamma_0 \cdot \Gamma_1 \ldots \Gamma_{D-1} = 1$$

and a second inequivalent basis is given by $-\Gamma_0$, Γ_J, $1 \le J \le D-1$.

Contrary to the embedding for even D described above, this embedding does not provide a direct mapping between spinor representations of the Lorentz group in D and $D-1$ dimensions, and we have to compensate for that through an extra factor $\mathcal{X}$ in the Kaluza–Klein Ansatz for spinors:

$$\Psi = \Phi^{\frac{1}{4(D-2)}} \mathcal{X} \cdot \psi,$$

$$\mathcal{X}^+ \cdot \Gamma_0 \cdot \mathcal{X} = \gamma_0,$$

$$\mathcal{X}^+ \cdot \Gamma_j \cdot \mathcal{X} = \Gamma_j, \qquad 1 \le j \le D-2.$$

In the bases of γ–matrices employed here, $\mathcal{X}$ is realized explicitly as

$$\mathcal{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Employing again a gauge $E^\mu{}_{D-1} = 0$ for the D–*bein* the reduced action reads

$$\sqrt{-G}\,\overline{\Psi}[E^M{}_A \Gamma^A (i\partial_M + i\Omega_M + qA_M) - M]\Psi = \tag{25}$$

$$= \sqrt{-g}\,[\overline{\psi} e^\mu{}_a \gamma^a (i\partial_\mu + i\omega_\mu + qV_\mu)\psi] - iq\sqrt{-g}\,\Phi^{-\frac{1}{2}}\overline{\psi}\gamma_D A\psi$$

$$-M\sqrt{-g}\,\Phi^{-\frac{1}{2(D-2)}}\overline{\psi}\psi - \frac{1}{8}\sqrt{-g}\,\Phi^{\frac{1}{2}} f_{ab}\overline{\psi}\gamma^{ab}\gamma_D\psi$$

with Ω and ω denoting the canonical spin connections for the metrics G and g, respectively, and again $V_\mu = A_\mu - a_\mu A$.

Since time reversal in odd dimensions mixes the two equivalence class of representations of the corresponding Clifford algebra, one has to add a second spinor Ψ_- related to $\Gamma^0_- = -\Gamma^0$, $\Gamma^J_- = \Gamma^J$ and a corresponding low–dimensional spinor ψ_- through

$$\Psi_- = \Phi^{\frac{1}{4(D-2)}} \mathcal{X}_- \cdot \psi_-$$

with

$$\mathcal{X}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

This yields then the same low–dimensional action with the sign of the parity violating terms inverted. To end up with an irreducible representation of the Lorentz group in $D-1$ dimensions we impose the constraint $\psi_- = \psi$, whence our $(D-1)$–dimensional action is

$$\mathcal{L} = \sqrt{-g}\,[\overline{\psi} e^\mu{}_a \gamma^a (i\partial_\mu + i\omega_\mu + qV_\mu)\psi] - M\sqrt{-g}\,\Phi^{-\frac{1}{2(D-2)}}\overline{\psi}\psi.$$

Both for even and odd values of D we have not encountered a photon–like coupling of the graviphoton to low–dimensional fermions. This is can be understood very easily: The only place where a photon–type coupling of the graviphoton could arise on zero modes of ∂_{D-1} is in the reduction of the spin–connection, through terms containing

$$E^{D-1}{}_a = -\Phi^{\frac{1}{2(D-2)}} a_\mu e^\mu{}_a.$$

However, all these terms contain derivatives on Φ, $e_\mu{}^a$ or a_μ and can not create a $U(1)$ gauge coupling. On the zero modes components of the metric tensor in higher dimensions modify gauge fields upon compactification, but they do not create new gauge couplings to fermions beyond those couplings already present in higher dimensions. Of course, a possible way to avoid this negative verdict on Kaluza–Klein generated gauge fields relies on eigenspinors of the internal Dirac operator $E^{D-1}{}_a \gamma^a \partial_{D-1}$, but this will play no role in the sequel.

In concluding this section I would like to add a remark on the string frame in four dimensions: String theory in the string frame supposes that the dilaton couplings in four dimensions look like Kaluza-Klein couplings with the four–dimensional metric directly induced from an embedding space, i.e. it is based on the strong rather than the weak version of Kaluza–Klein theory. Neglecting the gauge fields and denoting by $\Phi^2_{\mathcal{BD}}$ the determinant of the internal metric, compactification from D dimensions to the Einstein frame and a string–like frame in four dimensions would proceed via

$$G_{MN} = \Phi^{-1}_{\mathcal{BD}} \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \Phi^{\frac{D-2}{D-4}}_{\mathcal{BD}} h_{mn} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{\mu\nu} & 0 \\ 0 & \Phi^{\frac{2}{D-4}}_{\mathcal{BD}} h_{mn} \end{pmatrix},$$

respectively, and yield

$$\begin{aligned} \sqrt{-G}\,G^{MN}R_{MN} &= \sqrt{-g}\,g^{\mu\nu}(R_{\mu\nu} - \frac{D-2}{2(D-4)}\partial_\mu \ln \Phi_{\mathcal{BD}} \cdot \partial_\nu \ln \Phi_{\mathcal{BD}}) \\ &= \sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\Phi_{\mathcal{BD}}(\tilde{R}_{\mu\nu} + \frac{D-5}{D-4}\partial_\mu \ln \Phi_{\mathcal{BD}} \cdot \partial_\nu \ln \Phi_{\mathcal{BD}}). \end{aligned} \tag{26}$$

Therefore, in strong Kaluza–Klein theory gravity would be described by a Brans–Dicke type theory with a Brans–Dicke parameter

$$\omega_{\mathcal{BD}} = -\frac{D-5}{D-4}.$$

This is not yet gravity in the string frame, since both $\Phi_{\mathcal{BD}}$ and the string dilaton ϕ_S couple to the Yang–Mills terms, and the metric $\tilde{g}_{\mu\nu}$ has to be Weyl rescaled by $\exp(\sqrt{\frac{\kappa}{2}}\phi_S)$ to ensure equal coupling both to R and to F^2. The resulting theory in the gravitational sector is a Brans–Dicke theory (1) with

$$\omega_{\mathcal{BD}} = -1,$$

and initially $\Lambda(\Phi_{\mathcal{BD}}) = 0$. However, Brans–Dicke theory is constrained by the fact that solar system tests of gravity restrict Brans–Dicke parameters for massless Brans–Dicke scalars to $\omega_{\mathcal{BD}} > 500$ [111]. This leaves two possibilities for the string frame: We either should identify a mechanism to generate a large mass for $\Phi_{\mathcal{BD}}$ if low energy gravity in string theory is described by a scalar–tensor type theory in the string frame, or higher loop effects in string theory modify the Brans–Dicke coupling function from $R\Phi_{\mathcal{BD}}$ to a convex function $C(\Phi_{\mathcal{BD}})R$. In the second case the cosmological attractor mechanism of Damour and Nordtvedt would apply [28], and cosmological evolution would restore effective Einstein gravity in the string frame.

Our main interest in the present work is in light dilatons, and therefore we rely on the conservative assumption that low energy gravity is described by Einstein gravity.

Compactification of heterotic string theory in the Einstein frame shows that the four–dimensional dilaton ϕ coupling to gauge fields arises as a linear combination of the string dilaton ϕ_S already present in the ten–dimensional field theory limit and a Kaluza–Klein dilaton ϕ_K arising from the compactification to four dimensions [112]. If both ϕ_S and ϕ_K are normalized to have standard kinetic terms in four dimensions the dilaton is dominated by the Kaluza–Klein component with a mixing angle $\theta_\phi = -\frac{\pi}{6}$:

$$\phi = \frac{1}{2}(\sqrt{3}\phi_K - \phi_S),$$

and its decay constant is

$$f_\phi = \frac{m_{Pl}}{\sqrt{2}}. \tag{27}$$

This relies on a parametrization for the string dilaton such that strong gauge coupling in ten dimensions corresponds to strongly coupled string theory and is based on the fact that the ten–dimensional field theory limit of heterotic string theory consists of $N = 1$ supergravity coupled to $N = 1$ supersymmetric gauge theory [56, 58, 53]. This theory contains pieces derived from eleven–dimensional supergravity, but the string dilaton couples stronger than a Kaluza–Klein dilaton from eleven dimensions, and for this reason the dilaton

decay constant in four dimensions realizes a lower bound for Kaluza–Klein decay constants arising through compactifications from D dimensions:

$$f_{\phi(KK)} = \frac{m_{Pl}}{\sqrt{2}}\sqrt{\frac{D-2}{D-4}} > f_\phi.$$

The alert reader may wonder why neither in f_ϕ nor in any pure Kaluza–Klein decay constant any compactification scales show up. This is due to the fact that internal volumes rescale higher–dimensional gravitational constants to the four–dimensional constant κ, and in normalizing dilatons to standard kinetic terms only a rescaling with m_{Pl} is involved. Therefore the decay constants only depend on the Planck mass.

A dilaton coupling scale (27) of the order of the Planck mass implies invisibility of the dilaton from the particle physics point of view: We can readily calculate the integrated tree level cross section for creation of a dilaton pair through head–on collision of two gauge bosons with Mandelstam parameter s

$$\sigma = \frac{s}{64\pi f_\phi^4},$$

and this tells us that even for Planck scale collisions the cross section would be tiny $\sigma \simeq \frac{1}{4\pi} l_{Pl}^2$.

4 The dilaton in three dimensions

Interest in a three–dimensional dilaton arose from Witten's observation that theories with Bogomol'nyi saturated solitons may be related to theories in higher dimensions in a Kaluza–Klein type framework [113, 114]. This idea can be motivated from classical duality considerations which generically imply a trading between solitons and particles in dual theories. Given the soliton–particle correspondence and the infinite tower of equidistant solitonic excitations it seems very natural to relate solitons with an equidistant mass spectrum to compactified theories in higher dimensions. In this framework the particular case of dualities between supersymmetric theories in 2+1 dimensions and non–supersymmetric theories in 3+1 dimensions deserves special attention, since the four–dimensional theory might inherit the vanishing of the cosmological constant from the corresponding three–dimensional theory[6] [114, 115]. A discussion of the supersymmetric abelian Higgs model in 2+1 dimensions coupled to supergravity confirmed this picture by showing that the soliton spectrum in this theory is not supersymmetric [11]. It may also be worth–while to point out that due to the topological nature of gravity in three dimensions one would not need a fully fledged supergravity multiplet to get rid of the cosmological constant in this scenario. In particular, we would not need a gravitino, which would be hard to accomodate in the four–dimensional theory.

However, soon after Witten's proposal worries arose that the static potential in 2+1 dimensions would imply a logarithmic divergence of the dilaton for any static source. In spite of that it turned out that the infrared singularity of the propagator actually suppresses fluctuations of the dilaton in three dimensions. The suppression of fluctuations works because fermions and adjoint scalars provide sources for the dilaton which differ in sign from the dilaton sources provided by the gauge fields arising from the four–dimensional metric and four–dimensional gluons. Finite energy or independence of the theory from an infrared regulator then implies that any local dilaton source has to be compensated by another dilaton source somewhere else, whence the dilaton vanishes asymptotically and the radius of the internal dimension would approach a value to be fixed by string theory.

The idea to get rid of a dilaton through duality symmetries between theories in different dimensions is very speculative, but the mechanism outlined here [33] provides an example for non–perturbative stabilization of a dilaton in a low energy effective theory different from the mechanisms outlined in section 6.

In order to make the observations outlined above quantitative, I will take a four–dimensional point of view and discuss the action of the three–dimensional dilaton arising through a Kaluza–Klein parametrization of Einstein–Yang–Mills theory in four dimensions.

We relate the dilaton ϕ to the metric coefficient Φ via

$$\Phi = \exp(\sqrt{8\kappa}\phi),$$

[6] Supersymmetry as a solution to the cosmological constant problem has been discussed in [119]. A very useful review and critical discussion of several attempts to solve the problem can be found in [104].

where the three–dimensional gravitational constant κ is the four–dimensional gravitational constant divided by the circumference of the compact dimension. After appropriate rescalings of the other fields and coupling constants, we infer the following action in three dimensions from the results of the previous section:

$$\frac{1}{\sqrt{-g}}\mathcal{L} = \frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi - \frac{1}{4}\exp(\sqrt{8\kappa}\phi)f_{\mu\nu}f^{\mu\nu}$$

$$-\frac{1}{4}\exp(\sqrt{2\kappa}\phi)(V_{\mu\nu}{}^k V^{\mu\nu}{}_k + 2\sqrt{2\kappa}V_{\mu\nu}{}^k A_k f^{\mu\nu} + 2\kappa A_k A^k f_{\mu\nu}f^{\mu\nu})$$

$$+\overline{\psi}_+ e^\mu{}_a\gamma_+^a(i\partial_\mu + i\omega_{+\mu} + qV_\mu)\psi_+ + \overline{\psi}_- e^\mu{}_a\gamma_-^a(i\partial_\mu + i\omega_{-\mu} + qV_\mu)\psi_-$$

$$+q\exp(-\sqrt{2\kappa}\phi)(\overline{\psi}_+ A\psi_+ + \overline{\psi}_- A\psi_-) + M\exp\Big(-\sqrt{\frac{\kappa}{2}}\phi\Big)(\psi_+^+\psi_- + \psi_-^+\psi_+)$$

$$-\frac{1}{2}\exp(-\sqrt{2\kappa}\phi)g^{\mu\nu}D_\mu A_k D_\nu A^k - \frac{i}{8}\sqrt{2\kappa}\exp(\sqrt{2\kappa}\phi)f_{ab}(\overline{\psi}_+\gamma_+^{ab}\psi_+ + \overline{\psi}_-\gamma_-^{ab}\psi_-).$$

In order to evaluate the effect of the infrared divergence of the electrostatic potential in 2+1 dimensions, we consider the energy of a static configuration with the fermions in stationary orbits, and in gauge $A_0 = a_0 = 0$ (complying with $E^\mu{}_3 = 0$, since we employ diffeomorphisms which are constant along the normal direction):

$$\frac{1}{\sqrt{-g}}\mathcal{H} = \frac{1}{2}g^{ij}\partial_i\phi\cdot\partial_j\phi + \frac{1}{4}\exp(\sqrt{2\kappa}\phi)(V_{ij}{}^k + \sqrt{2\kappa}A^k f_{ij})(V^{ij}{}_k + \sqrt{2\kappa}A_k f^{ij})$$

$$+\frac{1}{4}\exp(\sqrt{8\kappa}\phi)f_{ij}f^{ij} - \overline{\psi}_+ e^j{}_a\gamma_+^a(i\partial_j + i\omega_{+j} + qV_j)\psi_+ - \overline{\psi}_- e^j{}_a\gamma_-^a(i\partial_j + i\omega_{-j} + qV_j)\psi_-$$

$$-q\exp(-\sqrt{2\kappa}\phi)(\overline{\psi}_+ A\psi_+ + \overline{\psi}_- A\psi_-) - M\exp\Big(-\sqrt{\frac{\kappa}{2}}\phi\Big)(\psi_+^+\psi_- + \psi_-^+\psi_+)$$

$$+\frac{1}{2}\exp(-\sqrt{2\kappa}\phi)g^{ij}D_i A_k D_j A^k + \frac{i}{8}\sqrt{2\kappa}\exp(\sqrt{2\kappa}\phi)f_{ab}(\overline{\psi}_+\gamma_+^{ab}\psi_+ + \overline{\psi}_-\gamma_-^{ab}\psi_-),$$

which tells us that the graviphoton and the Yang–Mills fields yield positive sources for the dilaton, while the kinetic term of A provides a negative contribution. There is some ambiguity with regard to the contribution due to the fermions. However, on–shell the fermion contribution adds up to a positive term. Therefore, generically the fermions will contribute negative sources to the dilaton.

This appearance of positive and negative source terms for the dilaton apparently works for any value of D. In ten–dimensional low energy effective actions of superstring theory involving only $N = 1$ supergravity the dilaton couples only with one sign since eleven–dimensional supergravity does not contain elementary fermions and Yang–Mills fields, and the residual 3–form potential and graviphoton are set to zero.

However, the case $D = 4$ is peculiar due to the logarithmic IR divergence of the electrostatic potential in 2+1 dimensions. In a linear approximation the dilaton $\ln\Phi$ behaves

like a massless scalar field coupled to external sources, whence finiteness of energy requires a vanishing dilaton charge:

$$\int d^2\mathbf{x}\,\varrho(\mathbf{x}) = \int d^2\mathbf{x}\,\Big(\sqrt{\frac{\kappa}{2}} f_{ij}f^{ij} + \sqrt{\frac{\kappa}{8}}(V+\sqrt{2\kappa}Af)_{ij}{}^{k}(V+\sqrt{2\kappa}Af)^{ij}{}_{k} \tag{28}$$

$$-\sqrt{\frac{\kappa}{2}} g^{ij} D_i A^k \cdot D_j A_k + \sqrt{2\kappa} q(\overline{\psi}_+ A\psi_+ + \overline{\psi}_- A\psi_-)$$

$$+\sqrt{\frac{\kappa}{2}} M(\psi_+^+\psi_- + \psi_-^+\psi_+) + \frac{i}{4}\kappa f_{ab}(\overline{\psi}_+\gamma_+^{ab}\psi_+ + \overline{\psi}_-\gamma_-^{ab}\psi_-)\Big) = 0.$$

This property is similar to charge neutrality of the Coulomb gas in two dimensions and can be derived from conformal invariance of the partition function or independence of the length scale λ entering the definition of the electrostatic potential:

$$\Phi(\mathbf{x}) = \exp\left(\frac{\sqrt{2\kappa}}{\pi}\int d^2\mathbf{x}'\,\varrho(\mathbf{x}')\ln\frac{|\mathbf{x}-\mathbf{x}'|}{\lambda}\right).$$

Eq. (28) is equivalent to absence of a logarithmic singularity of the dilaton.

If one feels uncomfortable about the use of a linear approximation in this argument one may alternatively rely on independence of the perturbation theory on the scale λ. λ appears as an infrared cutoff if the static result is inferred from the retarded potential

$$G(t-t',\mathbf{x}-\mathbf{x}') = \frac{\Theta\left((t-t')-|\mathbf{x}-\mathbf{x}'|\right)}{2\pi\sqrt{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2}}.$$

The self–trapping mechanism encoded in (28) is superficially stable with respect to quantum effects, since calculation of the 1–loop effective potential in dimensional regularization yields no perturbatively generated effective potential. This is explained for four dimensions in appendix B. I would also like to point out that the physics behind (28) is more transparent than in the case of the Coulomb gas, since particles and antiparticles contribute in the same way to the dilaton: If a gauge boson excites a dilaton field the divergence of the resulting energy density implies pair production of adjoint scalars and fermions to restore an asymptotically vanishing dilaton. Clearly, M suppresses the production of fermions relative to adjoint scalars. Stated in another way: The adjoint scalar and light fermions screen the dilaton charge of the gauge bosons.

Compactification to three dimensions and subsequent decompactification due to BPS solitons is an interesting, but speculative proposal for the low energy sector of string theory, and we will concentrate on the four–dimensional dilaton in the sequel.

5 Generalized Coulomb potentials in gauge theory with a dilaton

As pointed out in the previous sections we expect dilatonic degrees of freedom in four–dimensional gauge theories if physics at very high energies involves decompactification of internal dimensions or string theory. To acquire a better understanding of the impact of dilatons in four–dimensional gauge theory we now look into the problem how a light dilaton modifies the Coulomb potential and its non–abelian analog [36]. It turns out that the dilaton introduces an ambiguity due to different boundary conditions which can be imposed on the dilaton: Two interesting solutions which arise include a regularized potential proportional to $(r + r_\phi)^{-1}$, where r_ϕ is inverse proportional to the decay constant of the dilaton, and a confining potential proportional to r.

Here we are interested in low energy gauge theories, i.e. in the dynamics of initially massless modes from the point of view of string theory. Since the compactification scale or string scale are many orders of magnitude larger than the weak scale, where the low energy degrees of freedom described in the standard model of particle physics acquire their masses, we do not expect the dilaton to couple to the relevant masses at the weak scale. Modulo an effective potential which the dilaton may have acquired on the road down from the string/compactification scales to temperatures below the SUSY scale, the influence of a dilaton on a low energy gauge theory is then described by a Lagrange density

$$\mathcal{L} = -\frac{1}{4}\exp(\frac{\phi}{f_\phi})F_{\mu\nu}{}^j F^{\mu\nu}{}_j - \frac{1}{2}\partial^\mu\phi\cdot\partial_\mu\phi + \sum_{f=1}^{N_f}\overline{\psi}_f(i\gamma^\mu\partial_\mu + q\gamma^\mu A_\mu{}^j X_j - m_f)\psi_f, \tag{29}$$

with X_j denoting a defining N_c–dimensional representation of su(N_c).

I already set the axion to zero, since the static pointlike source considered below does not excite the axion field.

The equations of motion are

$$\partial_\mu\Big(\exp(\frac{\phi}{f_\phi})F^{\mu\nu}{}_i\Big) + q\exp(\frac{\phi}{f_\phi})A_\mu{}^j f_{ij}{}^k F^{\mu\nu}{}_k = -q\overline{\psi}\gamma^\nu X_i\psi, \tag{30}$$

$$\partial^2\phi = \frac{1}{4f_\phi}\exp(\frac{\phi}{f_\phi})F_{\mu\nu}{}^j F^{\mu\nu}{}_j, \tag{31}$$

$$(i\gamma^\mu\partial_\mu + q\gamma^\mu A_\mu{}^j X_j - m)\psi = 0, \tag{32}$$

where here and in the sequel flavor indices are suppressed.

To analyze eq. (30) we will find it convenient to rewrite it in terms of the chromo–electric and magnetic fields $E_i = -F_{0i}{}^j X_j$, $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}{}^l X_l$:

$$\nabla\cdot\Big(\exp(\frac{\phi}{f_\phi})\mathbf{E}\Big) - iq\exp(\frac{\phi}{f_\phi})(\mathbf{A}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{A}) = \varrho,$$

$$\partial_0\Big(\exp(\frac{\phi}{f_\phi})\mathbf{E}\Big) - \nabla\times\Big(\exp(\frac{\phi}{f_\phi})\mathbf{B}\Big) + iq\exp(\frac{\phi}{f_\phi})([\Phi,\mathbf{E}] + \mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{A}) = -\mathbf{j}$$

$$\partial_0\mathbf{B} + iq[\Phi,\mathbf{B}] + \nabla\times\mathbf{E} - iq(\mathbf{A}\times\mathbf{E} + \mathbf{E}\times\mathbf{A}) = 0,$$

$$\nabla\cdot\mathbf{B} - iq(\mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A}) = 0,$$

where in the gauge theory above $\varrho = q(\psi^+\cdot X_k\cdot\psi)X^k$, $j_i = q(\overline{\psi}\cdot\gamma_i X_k\cdot\psi)X^k$, and we have included the Bianchi identities. In this section we use the letter Φ for A^0.

To discuss the impact of the dilaton on the Coulomb potential we consider static configurations: $\partial_0\varrho = 0$, $\mathbf{j} = 0$. Then we learn from $\partial_\mu j^\mu - iq[A_\mu, j^\mu] = 0$ that Φ and ϱ are in the same Cartan subalgebra: $[\Phi,\varrho] = 0$.

Pointlike stationary charge distributions, which in the present setting give rise to the generalized Coulomb potentials, are special cases of $\mathrm{SU}(N_c)$ currents of the form

$$j^\mu(x) = \varrho^i(\mathbf{r})X_i\eta^\mu{}_0 = \rho(\mathbf{r})C^iX_i\eta^\mu{}_0 \tag{33}$$

carrying the same $\mathbf{r}$–dependence along any direction in color space. Such distributions arise for separable quark wave functions $\psi(x) = \varphi(x)\zeta$, where ζ is a constant Lorentz scalar in a spinor representation of $\mathrm{SU}(N_c)$, and $\varphi(x)$ is a $\mathrm{SU}(N_c)$–invariant Dirac spinor whose left and right handed components differ only by a phase. We also assume both factors normalized according to $\int d^3\mathbf{r}\,\overline{\varphi}\cdot\varphi = 1$, $\zeta^+\cdot\zeta = 1$.

For $\mathrm{SU}(N_c)$ charges of the form (33) the vector potential can consistently be neglected, whence $\mathbf{E} = -\nabla\Phi$ and the Yang–Mills equations reduce to

$$\nabla\cdot\Big(\exp(\frac{\phi}{f_\phi})\nabla\Phi\Big) = -\varrho,$$

$$[\Phi,\nabla\Phi] = 0.$$

Due to (33) the second equation is fulfilled as a consequence of the first equation.

Our aim is to determine the chromo–electric potential for a point charge

$$\varrho_i(\mathbf{r}) = qC_i\delta(\mathbf{r})$$

where C_i denotes the expectation value of the generator X_i in color space. From the relation

$$(X_i)_{ab}(X^i)_{cd} = \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{1}{2N_c}\delta_{ab}\delta_{cd} \tag{34}$$

one finds for arbitrary color content

$$\sum_{i=1}^{N_c^2-1} C_i^2 = \frac{N_c-1}{2N_c}.$$

We thus want to determine the field of a stationary pointlike quark from

$$\nabla\cdot\Big(\exp(\frac{\phi(\mathbf{r})}{f_\phi})\mathbf{E}_i(\mathbf{r})\Big) = qC_i\delta(\mathbf{r}), \tag{35}$$

$$\nabla \times \mathbf{E}_i(\mathbf{r}) = 0, \tag{36}$$

and

$$\Delta\phi(\mathbf{r}) = -\frac{1}{2f_\phi}\exp(\frac{\phi(\mathbf{r})}{f_\phi})\mathbf{E}_i(\mathbf{r})\cdot\mathbf{E}^i(\mathbf{r}). \tag{37}$$

The unique radially symmetric solution to (35) can be written down immediately:

$$\exp(\frac{\phi(r)}{f_\phi})\mathbf{E}_i(\mathbf{r}) = \exp(\frac{\phi(r)}{f_\phi})E_i(r)\mathbf{e}_r = \frac{qC_i}{4\pi r^2}\mathbf{e}_r \tag{38}$$

whence equation (36) is also satisfied. Equation (37) then translates into

$$\frac{d^2}{dr^2}\phi(r) + \frac{2}{r}\frac{d}{dr}\phi(r) = -\frac{q^2}{64\pi^2 f_\phi}\Big(1-\frac{1}{N_c}\Big)\exp\Big(-\frac{\phi(r)}{f_\phi}\Big)\frac{1}{r^4}. \tag{39}$$

The form of this equation suggests an ansatz $\frac{\phi(r)}{f_\phi} = a\ln(\frac{r}{b})$, which yields the solution discussed below. However, we can solve (39) for arbitrary boundary conditions through a substitution

$$\xi = \frac{q}{4\pi f_\phi r}\sqrt{\frac{1}{2}-\frac{1}{2N_c}}, \qquad \theta(\xi) = \frac{\phi(r)}{f_\phi}, \tag{40}$$

yielding[7]

$$\frac{d^2}{d\xi^2}\theta(\xi) = -\frac{1}{2}\exp(-\theta(\xi)), \tag{41}$$

or in terms of boundary conditions at infinity:

$$\theta'(\xi)^2 - \theta'(0)^2 = \exp(-\theta(\xi)) - \exp(-\theta(0)), \tag{42}$$

$$\xi = \int_{\theta(0)}^{\theta(\xi)} \frac{d\theta}{\sqrt{\exp(-\theta) - \exp(-\theta(0)) + \theta'(0)^2}},$$

where a sign ambiguity has been resolved by the requirement that the dilaton should not diverge at finite radius. The integral can be done elementary, with two branches depending on the sign of $\theta'(0)^2 - \exp(-\theta(0))$.

[7] We can map the dilaton equation of motion for arbitrary number d of spatial dimensions to eq. (41) through the substitution

$$\xi = \frac{q}{f_\phi}\sqrt{\frac{1}{2}-\frac{1}{2N_c}}G_d(r)$$

with

$$G_d(r) = -\frac{1}{2\pi}\ln(\frac{r}{r_0}), \qquad d=2,$$

$$G_d(r) = \frac{\Gamma(\frac{d}{2})}{2(d-2)\sqrt{\pi}^d}\frac{1}{r^{d-2}}, \qquad d>2.$$

The presence of the dilaton introduced a two–fold ambiguity in the Coulomb problem, and we have to determine from physical requirements which boundary conditions to chose.

For a first solution we require that the dilaton generated by the pointlike quark vanishes at infinity while the gradient satisfies the minimality condition

$$\lim_{r\to\infty} r^2 \frac{d}{dr}\phi(r) = -\frac{q}{4\pi}\sqrt{\frac{1}{2}-\frac{1}{2N_c}}. \tag{43}$$

This gives minimal kinetic energy for the dilaton at infinity subject to the constraint that the chromo–electric field does not develop a singularity for positive finite r. Then we find for the radial dependence of the dilaton and the electric field

$$\phi(r) = 2f_\phi \ln\Big(1+\frac{q}{8\pi f_\phi r}\sqrt{\frac{1}{2}-\frac{1}{2N_c}}\Big), \tag{44}$$

$$\mathbf{E}_i(\mathbf{r}) = \frac{qC_i}{4\pi\Big(r+\frac{q}{8\pi f_\phi}\sqrt{\frac{1}{2}-\frac{1}{2N_c}}\Big)^2}\mathbf{e}_r, \tag{45}$$

implying a modified Coulomb potential

$$\Phi_i(r) = \frac{qC_i}{4\pi r+\frac{q}{2f_\phi}\sqrt{\frac{1}{2}-\frac{1}{2N_c}}}. \tag{46}$$

The result for gauge group U(1) is received through the substitution $N_c \to -1$, and the corresponding dilaton–photon configuration was proposed already as a solitonic solution in a remarkable paper by Cvetič and Tseytlin [25].

The removal of the short distance singularity in the chromo–electric field would imply finite energy of the dilaton–gluon configuration:

$$E = \int d^3\mathbf{r}\Big(\frac{1}{2}\nabla\phi\cdot\nabla\phi+\frac{1}{2}\exp(\frac{\phi(\mathbf{r})}{f_\phi})\mathbf{E}_i(\mathbf{r})\cdot\mathbf{E}^i(\mathbf{r})\Big) = 2qf_\phi\sqrt{\frac{1}{2}-\frac{1}{2N_c}}. \tag{47}$$

This regularization of the Coulomb potential at high energies is a very attractive property: In view of the prediction (27) it means that the dilaton resolves pointlike singularities at the Planck scale, fitting very well with the interpretation of the dilaton as a low energy imprint of a theory of quantum gravity.

It is clear that even a dilaton photon coupling and a resulting regularization of the electromagnetic Coulomb potential at or below l_{Pl} would not contradict accelerator based "confirmations" of pointlike structure of electrons above 10^{-3} fm $\simeq 10^{16}\, l_{Pl}$, and similarly the dilaton would also not show up spectroscopically: Applied to the hydrogen atom, the regularization (46) would imply a shift of energy levels of order

$$\frac{\Delta E}{E} \simeq -\sqrt{\frac{\alpha_{em}}{8\pi}}\frac{l_{Pl}}{a_B} \simeq -2.6\times 10^{-28}.$$

For comparison, the 1S–2S level splitting and the Rydberg constant are known with a relative precision of order 10^{-11} [4], and counted in terms of orders of magnitude a dilaton regularized electromagnetic Coulomb potential is as invisible in high energy physics experiments as it is spectroscopically.

However, besides (46) there exists another quite intriguing solution if we require that $\frac{1}{q^2}\exp(\frac{\phi}{f_\phi})$ is independent of q. This requirement arises naturally in string theory, since the non–perturbatively fixed expectation value of the dilaton itself is supposed to determine the coupling. In the action (29) this requirement amounts to the constraint that the solution should respect the scale invariance of the equations of motion under

$$\phi \to \phi + 2\eta f_\phi$$

$$A \to \exp(-\eta)A$$

$$q \to \exp(\eta)q$$

for constant η. Eqs. (40,42) then imply $\theta'(\xi)^2 = \exp(-\theta(\xi)) = 4\xi^{-2}$, yielding

$$\phi(r) = 2f_\phi \ln\Big(\frac{q}{8\pi f_\phi r}\sqrt{\frac{1}{2} - \frac{1}{2N_c}}\Big), \tag{48}$$

$$\mathbf{E}_i(\mathbf{r}) = \frac{32\pi f_\phi^2}{q}\frac{N_c}{N_c - 1}C_i\mathbf{e}_r. \tag{49}$$

This corresponds to an energy density

$$\mathcal{H}(\mathbf{r}) = 4\frac{f_\phi^2}{r^2}$$

whence the energy in a volume of radius r diverges linearly:

$$E|_r = 16\pi f_\phi^2 r.$$

This is an infrared divergence, whence it should not be related to new physics at short distances, and it would cost an infinite amount of energy to create an isolated quark. Gauge theory with a dilaton thus accomodates both Coulomb and confining phases in a simple way.

6 The axidilaton and stabilization of the dilaton in four dimensions

Yet we have been missing the pseudo–scalar axion which couples to the instanton density $F \cdot \tilde{F}$. The motivation for including an axion in theories with a dilaton is four–fold: Historically the first and still a very important motivation for the axion arose from the observation that it explains the absence of a CP violating phase in gauge theories [77, 103, 110]. Besides this an axion arises also as a massless excitation of closed superstrings as a companion of the graviton and the dilaton [53], and it accompanies the dilaton in supersymmetric gauge theories: If the dilaton arises in the real part of the lowest component of a chiral superfield the corresponding imaginary part is an axion. Furthermore, under a certain constraint on decay constants the axion–dilaton system exhibits a duality symmetry commonly denoted as S–duality. This has been realized both in field theory [88] and in string theory[8] [47, 84, 85, 87]. The dilaton and the axion are mixed under this symmetry in a non–linear way, and recent developments in string theory indicate that S–duality should be a generic feature of grand unified quantum field theories inherited from string theory.

The primary motivations for contemporary considerations of a dilaton arise from string theory, and in this spirit emphasis in the present discussion will also be on a string inspired axion. The difference does not show up in the coupling to gauge fields, but in the vacuum sector: A Peccei–Quinn type axion is an angular variable and has at most finitely many different vacua. The string axion on the other hand arises as a dual field to an antisymmetric tensor and has no reason to be periodic. It also will not necessarily couple to light fermion masses, yet it still suppresses a CP violating θ–angle in non–abelian gauge theory. To explain how this comes about, note that an axion explains absence of CP violation with or without Peccei–Quinn symmetry in the fermionic sector: If we temporarily include the axion scale in the axion $\Theta(x)$, which thus becomes dimensionless, the relevant axion–gluon term in the presence of θ is

$$\frac{q^2}{32\pi^2}(\Theta + \theta) F^{\mu\nu}{}_j \tilde{F}_{\mu\nu}{}^j.$$

However, instantons always induce an effective axion potential such that the vacuum expectation values of the axion satisfy $\langle\Theta\rangle + \theta = 2\pi n$ for some integer n, and this eliminates CP violation from the $F\tilde{F}$–term. From this point of view Peccei–Quinn symmetry on the fermions only introduces an additional Higgs field, in order to derive the relevant interaction indirectly through an anomaly.

In the spirit of employing Kaluza–Klein theory as a paradigm for theories with a dilaton we first consider the axion from a five–dimensional point of view: In the presence of a five–dimensional threshold the axion should arise as the fifth component of a pseudo–vector.

[8] Target space duality mixes dilaton– and axion–like degrees of freedom in string theory in a similar manner, see [45, 46] and references there.

The Kaluza–Klein Ansatz (21) then yields

$$\frac{q^2}{64\pi^2}\sqrt{-G}\epsilon^{JKLMN}\Theta_J F_{KL}{}^j F_{MNj} = \frac{q^2}{64\pi^2}\sqrt{-g}\epsilon^{\mu\nu\rho\sigma}[\Theta F_{\mu\nu}{}^j F_{\rho\sigma j} + 4\Theta_\mu F_{\nu\rho j} D_\sigma A^j],$$

where $\epsilon_{01234} = -\sqrt{-G}$, i.e. ϵ is a tensor, not a density. Since this term transforms into a divergence under $\Theta_K \to \Theta_K + \partial_K \alpha$, it is natural to propose a Maxwell term as kinetic term:

$$-\frac{1}{4}\sqrt{-G}\,\Theta_{MN}\Theta^{MN} = -\frac{1}{4}\sqrt{-g}\,\Phi^{\frac{1}{3}}[b_{\mu\nu}b^{\mu\nu} + \frac{2}{\Phi}g^{\mu\nu}\partial_\mu\Theta\cdot\partial_\nu\Theta], \tag{50}$$

where $\Theta_{MN} = \partial_M\Theta_N - \partial_N\Theta_M$ and

$$b_{\mu\nu} = \Theta_{\mu\nu} + \sqrt{2\kappa}a_\mu\partial_\nu\Theta - \sqrt{2\kappa}a_\nu\partial_\mu\Theta.$$

Here we have rescaled the graviphoton $a_\mu \to \sqrt{2\kappa}a_\mu$ to have canonical mass dimension.

It is an interesting property of a five–dimensional threshold that the power of the dilaton in front of the kinetic term of the axion Θ is such that it matches exactly with the SL(2,$\mathbb{R}$) duality for the axidilaton system described below (55). This is a unique property of reductions from five to four dimensions and in remarkable coincidence with expectations from string theory.

In the sequel we will use the following pseudo–vector in four dimensions:

$$\overline{\Theta}_\mu = \Theta_\mu - \sqrt{2\kappa}a_\mu\Theta,$$

$$\overline{\Theta}_{\mu\nu} = b_{\mu\nu} - \sqrt{2\kappa}\Theta f_{\mu\nu}.$$

In order to motivate the following investigations, we take a closer look at the action of the zero modes of five–dimensional Einstein–Yang–Mills theory with fermions compactified to four dimensions. We relate the dilaton ϕ to the metric coefficient Φ by

$$\Phi = \exp(\sqrt{6\kappa}\phi),$$

where the four–dimensional gravitational constant κ is the five–dimensional gravitational constant divided by the circumference of the compact dimension. After appropriate rescalings of the other fields and coupling constants, we infer the following action in four dimensions from the results of section 3:

$$\frac{1}{\sqrt{-g}}\mathcal{L} = \frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi - \frac{1}{4}\exp(\sqrt{6\kappa}\phi)f_{\mu\nu}f^{\mu\nu}$$

$$-\frac{1}{2}\exp\Big(-\sqrt{\frac{8\kappa}{3}}\phi\Big)g^{\mu\nu}(D_\mu A_j\cdot D_\nu A^j + \partial_\mu\Theta\cdot\partial_\nu\Theta)$$

$$-\frac{1}{4}\exp\Big(\sqrt{\frac{2\kappa}{3}}\phi\Big)[V_{\mu\nu}{}^j V^{\mu\nu}{}_j + \overline{\Theta}_{\mu\nu}\overline{\Theta}^{\mu\nu} + 2\sqrt{2\kappa}(V_{\mu\nu}{}^j A_j + \overline{\Theta}_{\mu\nu}\Theta)f^{\mu\nu} + 2\kappa(A_jA^j + \Theta^2)f_{\mu\nu}f^{\mu\nu}]$$

$$+\frac{q^2}{64\pi^2 f_a}\sqrt{-g}\epsilon^{\mu\nu\rho\sigma}\Big(V_{\mu\nu}{}^j + \sqrt{2\kappa}(a_\nu D_\mu A^j - a_\mu D_\nu A^j + A^j f_{\mu\nu})\Big)$$

$$\times\Big(\Theta[V_{\rho\sigma j}+\sqrt{2\kappa}(a_\sigma D_\rho A_j-a_\rho D_\sigma A_j+A_j f_{\rho\sigma})]+4(\overline{\Theta}_\rho+\sqrt{2\kappa}a_\rho\Theta)D_\sigma A_j\Big),$$

and each fermion species of mass M at the compactification scale contributes a term

$$\frac{1}{\sqrt{-g}}\mathcal{L}_f=\overline{\psi}e^\mu{}_a\gamma^a(i\partial_\mu+i\omega_\mu+qV_\mu)\psi-M\exp\Big(-\sqrt{\frac{\kappa}{6}}\phi\Big)\overline{\psi}\psi.$$

Again, as pointed out already in section 3, the graviphoton ensures invariance of the effective four–dimensional gauge fields under diffeomorphisms along the fifth dimension:

$$V_\mu{}^j=A_\mu{}^j-\sqrt{2\kappa}a_\mu A^j$$

$$V_{\mu\nu}{}^j=F_{\mu\nu}{}^j+\sqrt{2\kappa}(a_\mu D_\nu A^j-a_\nu D_\mu A^j-A^j f_{\mu\nu}).$$

The classical equations of motion of the system above are invariant under constant shifts of the dilaton

$$\phi\to\phi+c, \tag{51}$$

$$e^\mu{}_a\to\exp\Big(-\sqrt{\frac{\kappa}{6}}c\Big)e^\mu{}_a,$$

$$a_\mu\to\exp\Big(-\sqrt{\frac{2\kappa}{3}}c\Big)a_\mu,$$

$$V_\mu\to V_\mu,\qquad \overline{\Theta}_\mu\to\overline{\Theta}_\mu,$$

$$\psi\to\exp\Big(-\frac{1}{2}\sqrt{\frac{\kappa}{6}}c\Big)\psi,$$

$$A\to\exp\Big(\sqrt{\frac{2\kappa}{3}}c\Big)A,$$

$$\Theta\to\exp\Big(\sqrt{\frac{2\kappa}{3}}c\Big)\Theta,$$

since this just rescales the action according to

$$S\to\exp\Big(\sqrt{\frac{2\kappa}{3}}c\Big)S.$$

This symmetry can equivalently be formulated as a scaling symmetry on the co–ordinates, and the dilaton is often denoted as a Goldstone boson for dilatations. However, the symmetry is unbroken as long as the dilaton remains massless, and I prefer the modern designation of the dilaton as a flat direction.

The origin of the symmetry of the equations of motion under (51) is easily understood from the Kaluza–Klein origin of the action. The scale transformations are equivalent to a rescaling of the internal dimensions by a factor $\exp\Big(\sqrt{\frac{2\kappa}{3}}c\Big)$, and the equations of motion resulting from a Kaluza–Klein Ansatz do not carry any remembrance of the internal scale

since we neglected any massive modes related to ∂_5. Therefore, we should not expect a perturbatively generated dilaton potential, since the perturbative dynamics of the Kaluza–Klein zero modes only depends on the fluctuations of the internal dimensions through the dilaton, but not on their actual size. However, if there exist inherently four–dimensional effects in the low energy dynamics, then we might expect a non–perturbatively generated dilaton potential, since unwinding of internal dimensions would certainly conflict with inherently four–dimensional effects. A genuine four–dimensional effect is the appearance of instantons in gauge theories, and therefore we will concentrate on the issue whether instantons create a dilaton potential. Indeed, we will find that instantons create a dilaton mass, because generically instantons imply that a small dilaton would be energetically favored, while the axions push the dilaton to large values.

This nicely complies with ideas about duality symmetries between axions and dilatons: Instantons create an effective axion potential and we have emphasized before that there emerged much evidence in recent years for a duality symmetry between the axion and the dilaton, which is described in eqs. (53–55) below. A fundamental axion acquiring an effective potential through instanton mediated tunneling effects thus provides a very strong indication for a light dilaton acquiring an effective potential in a similiar vein.

From a Minkowski space point of view an instanton contribution to an effective dilaton potential may also be described as a gluon condensate, and given the no–go conjecture for a perturbative origin condensates provide a natural mechanism to generate terms in a dilaton potential. Besides a gluon condensate we may expect from the coupling to the kinetic energy of the axion a contribution from a condensate $\langle \partial a \cdot \partial a \rangle$ [34, 35], or from a gluino condensate [30, 38, 94], and we will take a very brief look at a gluino condensate in the next section. Recent discussions of contributions from gluino condensates can be found in [67], where the coupling of the chiral dilaton multiplet is re–examined, and in [14], where the dilaton is treated in the linear multiplet. The proposal of a self–dual coupling of the axidilaton to the gluons is reviewed in [75].

In the present section we will concentrate on the contribution from the axions and its implications. Axions provide an attractive new mechanism for dilaton stabilization since the exponents of the dilaton multiplying the gluon and axion terms differ in sign, and since non–trivial axion configurations provide suggestive explanations for a condensate $\langle \partial a \cdot \partial a \rangle$. This alternative proposal for generation of a dilaton potential implies a drastic change of scales: While a gluino condensate would be expected to generate a dilaton potential at the SUSY breaking scale around or above 1 TeV, the axion would stabilize the dilaton at the QCD scale around 1 GeV.

In the sequel the notation for the axion will be changed $\Theta \to a$, since graviphotons will be neglected and a is a more standard notation for the axion in four–dimensional field theory. The main players in the game are then the dilaton ϕ, the axion a and gauge fields A_μ with field strengths $F_{\mu\nu}$, and their mutual interactions before taking into account non–perturbative effects are governed by the Lagrangian

$$\frac{1}{\sqrt{-g}}\mathcal{L} = \frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi - \frac{1}{2}\exp(-2\frac{\phi}{f_\phi})g^{\mu\nu}\partial_\mu a\cdot\partial_\nu a \tag{52}$$

$$-\frac{1}{4}\exp(\frac{\phi}{f_\phi})F_{\mu\nu}{}^{j}F^{\mu\nu}{}_{j}+\frac{q^2}{64\pi^2 f_a}\epsilon^{\mu\nu\rho\sigma}aF_{\mu\nu}{}^{j}F_{\rho\sigma j}.$$

The dilaton–axion and dilaton–gluon couplings in (52) match in such a way that the system exhibits an SL(2,$\mathbb{R}$) duality symmetry, or S–duality for short, if the scales are related by

$$f_\phi=\frac{8\pi^2}{q^2}f_a. \tag{53}$$

The invariance of the equations of motion under the duality transformations is most conveniently described in terms of the axidilaton

$$z=\frac{1}{f_a}[a+if_\phi\exp(\frac{\phi}{f_\phi})] \tag{54}$$

and the symmetry is realized via

$$z'=\frac{a_{11}z+a_{12}}{a_{21}z+a_{22}},\qquad\qquad a_{11}a_{22}-a_{12}a_{21}=1, \tag{55}$$

$$F'_{\mu\nu}-i\tilde{F}'_{\mu\nu}=(a_{21}z+a_{22})(F_{\mu\nu}-i\tilde{F}_{\mu\nu}),$$

which means that the self–dual part of the Yang–Mills curvature transforms like a half–differential on the axidilaton upper half–plane.

The invariance of equations of motion plus Bianchi identities is easily recognized if the equations of motion are written as

$$\frac{\partial^2 z}{(z-\bar{z})^2}-2\frac{\partial z\cdot\partial z}{(z-\bar{z})^3}=\frac{f_a}{32if_\phi^3}(F+i\tilde{F})^2,$$

$$D_\mu\mathrm{Im}\{z(F^{\mu\nu}{}_{j}-i\tilde{F}^{\mu\nu}{}_{j})\}=0,$$

where ∂^2 denotes the covariant Laplacian.

The scaling symmetry in (55) in the abelian case is similar to but different from the rescaling (51): $\delta a=2\varepsilon a$, $\delta\phi=2\varepsilon f_\phi$, $\delta A_\nu=-\varepsilon A_\nu$, and it implies a Noether current

$$\frac{1}{\sqrt{-g}}j^\mu_\phi=f_\phi g^{\mu\nu}\partial_\nu\phi+a\exp(-2\frac{\phi}{f_\phi})g^{\mu\nu}\partial_\nu a-\frac{1}{2}\exp(\frac{\phi}{f_\phi})F^{\mu\nu}A_\nu+\frac{q^2}{16\pi^2}\frac{a}{f_a}\tilde{F}^{\mu\nu}A_\nu. \tag{56}$$

Both in abelian and non–abelian theories the Peccei–Quinn symmetry $z\to z+a_{12}$ leaves the gauge potentials invariant and yields a conserved current

$$\frac{1}{\sqrt{-g}}j^\mu_a=f_a\exp(-2\frac{\phi}{f_\phi})g^{\mu\nu}\partial_\nu a+\frac{q^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}(A_\nu{}^{j}\partial_\rho A_{\sigma j}+\frac{q}{3}f_{ijk}A_\nu{}^{i}A_\rho{}^{j}A_\sigma{}^{k}), \tag{57}$$

and the scaling symmetry (51) of the equations of motion is also preserved with $2\kappa\to 3f_\phi^{-2}$.

In view of the currents (56,57) the designation of f_ϕ and f_a as decay constants looks very natural and suggestive, since the scales parametrize non–vanishing matrix elements

between the (pseudo–)scalars and the vacuum, similar to the pion decay constant. However, there is an important difference which should be kept in mind: The pion decay constant parametrizes matrix elements $\langle 0|j_\pi^\mu|\pi\rangle$ which actually contribute to pion decays into lepton pairs through intermediate vector bosons, and the matrix element arises in the microscopic theory at low energies when the leptonic sector of the 4–Fermi–vertex has already been evaluated. Nothing like that is expected to take place for a fundamental axidilaton in string theory or Kaluza–Klein theory, and the scales f_ϕ and f_a appear only with negative powers in physical matrix elements.

Yet we have not taken into account non–trivial field configurations of the gauge fields and the axion: We infer the non–perturbative effects of these field configurations from the Lagrangian of the Euclidean action. In a flat background this takes the form:

$$\mathcal{L}_E = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi + \frac{1}{2}\exp(-2\frac{\phi}{f_\phi}) g^{\mu\nu}\partial_\mu a\cdot\partial_\nu a \tag{58}$$

$$+\frac{1}{4}\exp(\frac{\phi}{f_\phi}) F_{\mu\nu}{}^j F^{\mu\nu}{}_j - i\frac{q^2}{32\pi^2 f_a} a\tilde{F}_{\mu\nu}{}^j F^{\mu\nu}{}_j.$$

Positivity of the real part and the estimate of the effective axion potential by Vafa and Witten [97] indicate that the dominating contributions to the path integral come from instanton configurations $F = \pm\tilde{F}$ with constant dilaton and the axion frozen to integer multiples of $2\pi f_a$. This survival of instantons in the presence of the dilaton is crucial, since integrality of the instanton number and invariance of the path integral discretize Peccei–Quinn symmetry

$$\frac{a_{12}}{2\pi} \in \mathbb{Z},$$

thereby also breaking the scale invariance (51).

The impact of instantons on the effective axion potential has been examined by several authors, and the interpretation of instantons as real time tunneling configurations between gauge theory vacua suggests

$$V(a) = m_a^2 f_a^2\Big(1 - \cos(\frac{a}{f_a})\Big) \tag{59}$$

if the instanton gas is dilute enough to neglect higher order cosine terms [23, 57, 63]. While initially this result was inferred from semiclassical calculations of tunneling amplitudes, the same potential can also be derived in a direct instanton calculation if the wavelength of the axion is large compared to the instanton size.

The picture emerging from this shows us that instantons create an effective axion potential with an enumerable set of equidistant vacua, thus discretizing Peccei–Quinn symmetry. Discreteness of the axion vacua and the cosine–like shape of the axion in turn breaks the scale invariance (51) and indicates that the axidilaton–gluon system also lifts the degeneracy of the dilaton. This is obvious in the gauge sector: Instantons push the dilaton into the strong gauge coupling regime, since the action of the instantons decreases with decreasing $\langle\phi\rangle$. However, non–trivial configurations also arise in the axion sector:

– If a is periodic $a \sim a + 2\pi f_a$, then it contributes non–trivial configurations to the Euclidean path integral over $\exp(-S_E)$ in the form of *axion walls* (instead of axion strings in three dimensions). Periodicity of a arises, if it is related to the argument of a complex field with frozen modulus in the low energy regime. This is e.g. the case if a arises as the phase of a determinant of local fermion masses.
– If a is not an angular variable, then all the possible vacua $\langle a \rangle = 2\pi f_a n$ are distinct and we expect three–dimensional domain walls separating four–dimensional domains where a approximates different vacua.

String theory is essential for the stability of these defects, since the four axion scattering amplitude at string tree level depends non–trivially on the momenta of the scattered axions [83], whence (52) contains only lowest order terms in a derivative expansion, as expected for an effective low energy theory. However, we will not attempt a systematic derivation of the higher order derivative terms from string scattering amplitudes, but rather adopt a phenomenological approach in borrowing methods from the theory of cosmic strings to estimate the axion condensate.

Both kinds of topological defects mark regions of non–vanishing gradients ∂a and favor large values of the dilaton through the dilaton–axion coupling, thus compensating the effect of the instantons. Therefore, we expect an effective dilaton potential cutting off large values of the dilaton through an average background field strength, while small values of the dilaton are suppressed by the variance of the axion. After adjustment of q the potential results:

$$V(\phi) = \frac{m_\phi^2 f_\phi^2}{6}\Big(2\exp(\frac{\phi}{f_\phi}) + \exp(-2\frac{\phi}{f_\phi})\Big). \tag{60}$$

In the resulting model the dilaton mass m_ϕ and the coupling constant q parametrize the background field strength from the instantons and the axion gradients from domain boundaries.

We may give estimates on q and m_ϕ in terms of an average instanton scale ϱ and a characteristic length Δ of the axion defects. In the case of an angular axion Δ would measure the circumference of the axion walls, while in the case of vacuum domains of the axion the four–dimensional domain boundaries are extended in three dimensions and have an average thickness Δ in the fourth direction. The average separation ϱ of instantons in the instanton liquid is about three times larger than the average extension of the instantons [90, 82]. From this we find an estimate for the effective dilaton potential

$$V(\phi) = \frac{16}{q^2\varrho^4}\exp(\frac{\phi}{f_\phi}) + 2\pi^2\frac{f_a^2}{\Delta^2}\exp(-2\frac{\phi}{f_\phi}). \tag{61}$$

This implies for the gauge coupling and the dilaton mass

$$\frac{1}{q} = \frac{\pi f_a \varrho^2}{2\Delta}, \tag{62}$$

$$m_\phi f_\phi = \frac{4\sqrt{3}}{q\varrho^2}. \tag{63}$$

This investigation can be pursued further if a is not an angular variable: In this case we may estimate the parameter Δ by minimizing the energy density of the axion domain boundaries

$$u = 2\pi^2 \frac{f_a^2}{\Delta} + m_a^2 f_a^2 \Delta \tag{64}$$

yielding a thickness of the order

$$\Delta \simeq \frac{\pi\sqrt{2}}{m_a} \tag{65}$$

which is of the same order as the thickness of ordinary axion domain walls in Minkowski space [64]. From (62) and (65) we find a relation between the axion parameters and the average instanton radius

$$m_a^2 f_a^2 \simeq \frac{2}{\pi \alpha_q \varrho^4}. \tag{66}$$

The average instanton radius at low temperatures is approximately [90]

$$\frac{1}{\varrho} \simeq 200\,\mathrm{MeV}$$

and the big uncertainty in estimating axion parameters from (66) concerns the coupling constant at the scale where instanton mediated tunneling induces the axion mass. In QCD the investigation of a truncated Dyson–Schwinger equation for the gluon propagator by Alkofer, Hauck and von Smekal indicates a value $\alpha_q \sim O(10)$ [2], and with the previous results this points at a value

$$m_a f_a \sim 10^4\,\mathrm{MeV}^2. \tag{67}$$

Although we did not use any information or assumptions about axion–fermion couplings, it is remarkable that (67) complies with current algebra based estimates for an axion coupling to light quarks [9, 62, 92, 51, 10]:

$$m_a f_a \sim m_\pi f_\pi.$$

The potential (61) and eq. (66) imply a relation between the dilaton mass and the axion mass

$$m_\phi f_\phi \simeq \sqrt{6} m_a f_a. \tag{68}$$

Since S–duality will certainly be broken at the scales under consideration we expect a dilaton decay constant of the order of the Planck mass while f_a could be in the phenomenologically preferred range $10^{10} - 10^{12}$ GeV. Compared to the axion this implies a much smaller mass of the dilaton and a later onset of coherent dilaton oscillations.

7 The supersymmetric theory

The supersymmetric framework is of interest both from up–down and bottom–up approaches to physics beyond the standard model of particle physics. On the one hand on an intermediate energy scale below the compactification scale string theory predicts that the relevant physical degrees of freedom should be described by a supergravity theory. On the other hand supersymmetric gauge theories evolved into a primary tool for model building beyond the standard model, and a footprint of supersymmetry is considered as one of the most spectacular results that might be expected from accelerator physics on foreseeable time scales. The subject got a further boost a few years ago by the approximate convergence of coupling constants around 10^{16} GeV if supersymmetry begins to apply at the TeV–scale[9] [44, 52, 3, 65].

Given the necessity to introduce an axion to solve the strong CP problem, supersymmetry provides an independent motivation to also introduce a dilaton, since an axion in a chiral superfield always comes with a dilaton [112, 31].

With a few notational changes we will follow the conventions of Wess and Bagger [106]. Supersymmetry is conveniently decribed in terms of superfields. These are Grassmann valued fields over space–time, where the Grassmann algebra is generated by a constant Dirac spinor θ of mass dimension $-\frac{1}{2}$:

$$\{\theta_\alpha, \theta_\beta\} = 0, \quad \{\theta_\alpha, \bar\theta_{\dot\beta}\} = 0, \quad \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0.$$

The particular superfields which we need are the chiral dilaton multiplet

$$S = d + i\theta\cdot\sigma^\mu\cdot\bar\theta\,\partial_\mu d + \frac{1}{4}\theta^2\bar\theta^2\partial^2 d + \sqrt{2}\theta\cdot\delta - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\delta\cdot\sigma^\mu\cdot\bar\theta + \theta^2 Z$$

and the spinorial chiral superfield containing the gauge fields:

$$W_\alpha = -i\lambda_\alpha + \theta\cdot\sigma^\mu\cdot\bar\theta\,\partial_\mu\lambda_\alpha - \frac{i}{4}\theta^2\bar\theta^2\partial^2\lambda_\alpha + \theta_\alpha D - \frac{i}{2}\theta^2\sigma^\mu{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu D$$

$$+\frac{i}{2}(\sigma^\mu\cdot\bar\sigma^\nu)_\alpha{}^\beta\theta_\beta F_{\mu\nu} + \frac{1}{4}\theta^2(\sigma^\mu\cdot\bar\sigma^\nu\cdot\sigma^\rho)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\rho F_{\mu\nu} + \theta^2\sigma^\mu{}_{\alpha\dot\alpha}(\partial_\mu\bar\lambda^{\dot\alpha} - iq[A_\mu, \bar\lambda^{\dot\alpha}]).$$

The lowest component d of the dilaton multiplet is related to the axidilaton (54) through

$$d = i\frac{q^2}{16\pi^2}\bar z,$$

and we have normalized the multiplet such that it has mass dimension zero, i.e. the dilatino δ has mass dimension $\frac{1}{2}$. Note that our gauge field A_μ and the gauge fields v_μ in [106] differ in sign. The gluino λ is the fermionic superpartner of the gluon.

[9] This analysis is continuously improved, and predictions for masses of supersymmetric particles are to a large degree model dependent [3, 66].

The dilaton gluon coupling arises as the real part of the θ^2–component of SW^2:

$$SW_j^\alpha W_\alpha^j|_{\theta^2} = -\lambda_j^\alpha \lambda_\alpha^j Z + i\sqrt{2}\delta^\alpha \lambda_\alpha^j D_j - \frac{1}{\sqrt{2}}(\sigma^\mu \cdot \bar{\sigma}^\nu)_\alpha{}^\beta F_{\mu\nu}{}^j \lambda_j^\alpha \delta_\beta \tag{69}$$

$$+d[D^j D_j + i(D_\mu \lambda)^j \cdot \sigma^\mu \cdot \bar{\lambda}_j - i\lambda_j \cdot \sigma^\mu \cdot (D_\mu \bar{\lambda})^j - \frac{1}{2}F_{\mu\nu}{}^j F^{\mu\nu}{}_j - \frac{i}{2}\tilde{F}_{\mu\nu}{}^j F^{\mu\nu}{}_j],$$

and Witten observed that the correct supersymmetrization of kinetic terms of the axidilaton is given by a logarithm of superfields [112]:

$$\ln(S+S^+)|_{\theta^2\bar{\theta}^2} = \frac{1}{(d+d^+)^2}(\partial d^+ \cdot \partial d + \frac{i}{2}\delta \cdot \sigma^\mu \cdot \partial_\mu \bar{\delta} - \frac{i}{2}\partial_\mu \delta \cdot \sigma^\mu \cdot \bar{\delta} - Z^+ Z). \tag{70}$$

Neglecting the quarks and squarks, the supersymmetrization of (52) is then given by

$$\mathcal{L} = -2f_\phi^2 \ln(S+S^+)|_{\theta^2\bar{\theta}^2} + \frac{1}{2}SW_j^\alpha W_\alpha^j|_{\theta^2} + \frac{1}{2}S^+ \overline{W}_{\dot{j}}^{\dot{\alpha}} \overline{W}_{\dot{\alpha}}^{\dot{j}}|_{\bar{\theta}^2} \tag{71}$$

if the decay constants satisfy the self–duality condition (53).

In passing we also note that there exist numerous possibilities to supersymmetrize the effective axion potential (59), e.g. through a superpotential

$$\mathcal{V} = 2m_a f_a^2 \cosh\Big(\frac{f_\phi}{f_a}(S - \frac{1}{2})\Big).$$

However, since $m_a f_a \ll \Lambda_{SUSY}^2$ instantons will not dominate the gauge theory vacuum above the scale of supersymmetry breaking, and we will not pursue these superpotentials further.

With eqs. (68) and (71) at hand we may now also provide an estimate on the lower bound of values of the dilaton decay constant for which our approximation of dominance of the axion condensate is applicable:

Elimination of the auxiliary field Z of the dilaton multiplet yields the dilaton gluon coupling

$$\mathcal{L}_{\lambda\phi} = -\frac{1}{8f_\phi^2}\exp(2\frac{\phi}{f_\phi})\bar{\lambda}^2\lambda^2.$$

If there is a gluino condensate with a scale $\Lambda_{SUSY} \sim 1$ TeV, this would contribute a dilaton mass term[10] $m_\phi^\lambda \simeq \Lambda_{SUSY}^3 f_\phi^{-2}$. As a consequence the variance of the axion would dominate the dilaton mass at scales where instanton induced tunneling becomes relevant for dilaton decay constants above 10^{11} GeV, while for decay constant below this value the gluino condensate would dominate to very low temperatures.

[10]Note that restoration of the dilaton decay constant shows that a dilaton mass from gluino condensates would be tiny, too. Therefore the dilaton would also provide a suitable candidate for cold dark matter in the framework of supersymmetry breaking.

8 The dilaton as a dark matter candidate

We have seen that both instanton tunneling and supersymmetry breaking favor a very light weakly coupled dilaton accompanying a light weakly coupled axion. However, light weakly coupled (pseudo–)scalars (or scalars for short) are generically expected to make an appreciable contribution to the energy density of the universe in the form of cold dark matter, which in this specific instance means that they developed coherent oscillations with non–relativistic momenta. The onset of coherent oscillations is expected when the universe has cooled down to temperatures where mass terms begin to dominate over dissipative expansion terms in the equations of motion of the scalars. Coherent oscillations are then expected to dominate the energy density of the scalars, since due to the weak coupling thermal creation and annihilation of the scalars can be neglected.

There exist several monographs where the general background for cosmology and the impact of particle physics is very well presented, see e.g. [15, 64, 79]. However, I will begin with a review of a few basic facts to set the stage for the discussion of the role the dilaton.

In discussing cosmological implications of the dilaton we will stick to the usual approximation of spatial homogeneity and isotropy, i.e. we will discuss dynamics in a Robertson–Walker space–time with line element

$$ds^2 = -dt^2 + \mathcal{R}^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2\Big), \tag{72}$$

where r is dimensionless and the scale factor $\mathcal{R}$ has the dimension of a length. For $k = 1$ the spatial part of the metric can be described as a 3–sphere of radius $\mathcal{R}(t)$ in flat $\mathbb{R}^4$, while $k = -1$ is the corresponding hyperbolic space. $k = 0$ is flat 3–space.

The form (72) of the metric implies that the matter energy momentum tensor has the form $T_{00} = \varrho(t)$, $T_{0j} = 0$, $T_{ij} = p(t)g_{ij}$, and that energy conservation $\nabla_\mu T^{\mu 0} = 0$ reads

$$\frac{d}{dt}(\varrho\mathcal{R}^3) = -p\frac{d}{dt}(\mathcal{R}^3). \tag{73}$$

If particle interactions are fast enough to maintain thermal equilibrium during expansion, the first and second law of thermodynamics in a system with N_i particles of chemical potential μ_i

$$dE = TdS - pdV + \sum_i \mu_i dN_i$$

implies due to eq. (73)

$$\frac{dS}{dt} = -\sum_i \frac{\mu_i}{T}\frac{dN_i}{dt}, \tag{74}$$

where S and N_i are the entropy and particle numbers in a unit of comoving volume $v = V\mathcal{R}^{-3}$.

I would like to add a remark on the thermodynamical expression for the energy density appearing in the energy momentum tensor, since there exists some confusion about this central object:

There are many ways to divide a thermodynamic potential by a volume, but the energy density T_{00} is

$$\varrho = \left.\frac{\partial E}{\partial V}\right|_t .$$

However, in a FRW universe temperatures and chemical potentials will only depend on t, while particle numbers also go with the volume. From this we may immediately translate the expression for ϱ into thermodynamics:

$$\varrho = \left.\frac{\partial E}{\partial V}\right|_{T,\mu} = T\left.\frac{\partial S}{\partial V}\right|_{T,\mu} - p + \mu \cdot \left.\frac{\partial N}{\partial V}\right|_{T,\mu},$$

with an obvious abbreviation for the sum over particle species. From the grand potential $\Omega_{GC}(T,V,\mu) = -pV$, $d\Omega_{GC} = -SdT - pdV - N \cdot d\mu$ we learn that

$$\left.\frac{\partial S}{\partial V}\right|_{T,\mu} = \left.\frac{\partial p}{\partial T}\right|_{V,\mu},$$

and that the particle densities are

$$\nu_i \equiv \left.\frac{\partial N_i}{\partial V}\right|_{T,\mu} = \left.\frac{\partial p}{\partial \mu_i}\right|_{T,V,\hat{\mu}_i},$$

where $\hat{\mu}_i = \mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots$. With $\beta = \frac{1}{kT}$ a useful expression for ϱ is then

$$\varrho = T\left.\frac{\partial p}{\partial T}\right|_{V,\mu} - p + \mu \cdot \nu = \mu \cdot \nu - \left.\frac{\partial}{\partial \beta}(\beta p)\right|_{V,\mu}, \tag{75}$$

because this directly relates ϱ to Ω_{GC}.

In the case of one particle species the dynamical evolution would then be determined as follows: The energy levels of the particles determine the grand potential and the pressure $p(T,\mu)$. Eq. (75) is then used to calculate $\varrho(T,\mu)$, while in the thermodynamic limit the entropy and particle number in a comoving volume are $S = \frac{\partial p}{\partial T}\mathcal{R}^3$, $N = \frac{\partial p}{\partial \mu}\mathcal{R}^3$. Equations (73,74) and the Friedmann equation (76) below then constitute a set of three first order differential equations for the dynamical variables $\mathcal{R}(t)$, $T(t)$ and $\mu(t)$

The algebra of dynamical degrees of freedom in a quasi–statically expanding FRW model is therefore generated by the scale parameter which describes expansion or contraction with a relative velocity $H = \frac{d}{dt}\ln(\mathcal{R})$, the temperature T and the chemical potentials of the various particle species. Of course, in practice equilibrium is not maintained for weakly interacting particle species which decouple due to thinning out in the expanding universe. This is taken care of by assigning extra effective temperatures governing the energy distribution within these particles species. Furthermore, it proved a very useful approximation to calculate the history of the homogeneous and isotropic background piecing together different epochs where the energy content of the universe was stored primarily either in radiation, in pressureless matter, or in scalar fields. In the first two cases the

relevant degrees of freedom for the evolution of the background metric are ϱ, p and $\mathcal{R}$, and eq. (74) is replaced by a dispersion relation $p = p(\varrho)$, while in the case of scalar fields the relevant degrees of freedom are the scalar fields and $\mathcal{R}$, and the evolution is governed by the equations of motion of the scalar fields and the Friedmann equation. The explicit matter content and interactions determine the sequence and transitions of epochs in this approximation, and the present epoch of dust dominance $p \simeq 0$ outnumbers all previous epochs since expansion of the very hot and dense primeval plasma in its duration $\sim 10^{17}$ seconds. When solutions of the Friedmann equation for $p > -\varrho$ are evolved backwards in time we inavoidably hit an initial singularity $\varrho \to \infty$, $\mathcal{R} = 0$ for finite parameter t, and we will stick to the usual terminology of initial singularity or big bang, although we can only be sure that there existed a very hot and dense phase at some very early stage of our contemporary epoch[11]. The Hubble parameter H_0 corresponding to the present value of $H(t)$ is still subject to a seminal debate among astronomers, and this is encoded in an uncertainty parameter h which varies between $0.5 < h < 0.85$ [76]:

$$H_0 = 100h \frac{\mathrm{km}}{\mathrm{Mpc} \cdot \mathrm{s}} = 1.02h \times 10^{-10} \frac{1}{\mathrm{yr}}.$$

It may be worthwhile to recall for the justification of (72) that the DMR experiment on COBE measured temperature anisotropies in the cosmic microwave background of order $\frac{\delta T}{T} \simeq 10^{-5}$ in multipole expansions up to $l = 30$, see [13, 117, 54] and references there, as well as [86] and [68] for compilations of measurements since COBE. The exciting result for the experts was the transition from an era of upper bounds on the anisotropy to actual measurements, but the results also show how good an approximation an isotropic universe represents up to the decoupling of the cosmic background radiation around 10^{12} seconds after the big bang[12].

Taking into account energy conservation (73), the Einstein equations reduce to

$$\dot{\mathcal{R}}^2 + k = \frac{\kappa}{3} \mathcal{R}^2 \varrho \tag{76}$$

and a flat universe would correspond to an averaged contemporary mass density

$$\varrho_c = \frac{3}{\kappa} H_0^2 = 1.9h^2 \times 10^{-26} \frac{\mathrm{kg}}{\mathrm{m}^3} = 81h^2 (\mathrm{meV})^4 = 2.4h^2 \times 10^{-120} m_{Pl}^4.$$

Following the usual habit among astronomers energy densities will be measured in units of ϱ_c in this section: $\Omega = \frac{\varrho}{\varrho_c}$.

[11] This qualification may seem strange, since our present investigation is mainly motivated from string theory. However, even within string theory we are currently in a phase of exploring more and more hitherto unknown (and unexpected) possibilities for the high energy sector of the theory, and yet there has not emerged a coherent proposal for the evolution of space–time near the Planck scale.

Besides this, quantum groups provide another set of challenging ideas about the shortest distance structure of space–time.

[12] Homogeneity is much more subtle from the experimental point of view, see the discussion in [15].

Particle physics influences (and is increasingly influenced by) cosmology in many ways. Two instances where scalar particles play major roles are inflation and dark matter:

Inflation denotes a phase of accelerated expansion of the universe when distances grew faster than light cones. This happens for pressure to density ratios between $-1 \leq \frac{p}{\varrho} < -\frac{1}{3}$, and temporary superluminal expansion has the potential to solve several major problems in cosmology, in particular the horizon problem, the neglegibility of contributions to Ω from topological defects, and the problem why the measured energy density is not far away from ϱ_c. These and other motivations for inflation are very thoroughly reviewed in [15, 54, 79].

Although inflation seems to become an integral part of ongoing extensions of the standard cosmological model, I will not address it any further, since I do not expect that the dilaton which we examine here provides a suitable candidate for the sought for inflaton: We will see that thermally produced dilatons contribute at most 2 percent to the energy density of the universe at the scales where instantons and axions are expected to induce a mass term. On the other hand, this mass term also implies that the dilaton field can store energy in coherent oscillations which can provide a considerable amount of the contemporary energy density. Contrary to radiatively stored energy this energy would only very slowly dissipate into thermal energy of non–relativistic matter. In discussing this we will rely on the conservative assumption that at the time of onset of oscillations most energy is still stored in relativistic matter, i.e. we will calculate in a given Friedmann–Robertson–Walker background expanding according to $\mathcal{R} \sim \sqrt{t}$. Then the oscillations behave like non–relativistic matter with their energy density decreasing according to $\varrho \sim \mathcal{R}^{-3}$. In an alternative scenario one might speculate that the axion or dilaton could trigger temporary superluminal expansion of the universe with approximately constant energy density, if the axion and/or dilaton field would dominate the energy density immediately after the onset of oscillations. However, this would require a yet unknown mechanism to convert most of the thermal energy in relativistic matter into coherent axion or dilaton fields, and it would conflict with the successful interpretation of the cosmic microwave background as a remnant of the hot radiatively dominated phase before recombination if the energy is not re–converted into radiation.

Contrary to the verdict about dilaton induced inflation in FRW backgrounds, it turns out that a light dilaton can very well contribute to dark matter in the universe: There is wide agreement in the astronomy/astrophysics community that a considerable fraction of the contemporary energy density of the universe must be due to non–luminous matter, and the problem is to determine the nature of this matter. It seems clear now that part of this matter is of non–baryonic origin, since even for the lowest possible values of Deuterium abundance[13] and a Hubble constant as low as $50\frac{\mathrm{km}}{\mathrm{Mpc{\cdot}s}}$ primordial nucleosynthesis allows for a maximal baryonic contribution to the energy density of order $\Omega_B \leq 0.08$ [96], while both galactic motion and velocities on larger scales indicate values of Ω beyond 0.1. The recent survey of large scale peculiar motion in the nearby universe of Strauss and Willick [93] gives a range $0.3 \leq \Omega \leq 1$. Large scale motion, structure formation through gravitational

[13]Low D abundance means large conversion into ^{4}He due to large baryon density.

attraction, and gravitational lensing also indicate that a considerable amount of dark matter is concentrated in halos around galaxies or groups of galaxies, whence a large fraction of the dark matter has to be cold. For very massive particles this means that they had to be non–relativistic when they decoupled due to thinning out in the expanding universe. Very weakly interacting light particles contribute to cold dark matter (CDM) through non–relativistic coherent oscillations, as has been pointed out before and will be explained below. From the particle physics point of view the leading contenders for the CDM component of dark matter are the axion [64, 63], the lightest supersymmetric particle commonly denoted as a neutralino, and more recently the dilaton [50, 28, 34, 35, 49].

In this section we will focus on a dilaton whose mass at very low temperatures is dominated through instanton effects and discuss evolution of the axidilaton in an expanding universe. For this purpose we will concentrate on a temperature range between 1 TeV and 100 MeV, since we expect that a light axidilaton acquires its masses in that range (for $10^{11}\text{GeV} \leq f_\phi \leq 10^{18}\text{GeV}$) and it makes sense to assume that besides our hypothesized axidilaton no further degrees of freedom beyond the standard model will be relevant at these scales.

The universe has cooled down to temperatures 1 TeV and 100 MeV around $10^{-13}-10^{-12}$ and $10^{-5}-10^{-4}$ seconds after the initial singularity[14]. In this energy range the universe is radiation dominated with relativistic background matter satisfying a dispersion relation $\varrho = 3p$. The density and the scale factor then evolve according to

$$\varrho(t) = \frac{3}{4\kappa t^2}\frac{(\mathcal{R}_0^2 + kt_0^2)^2}{(\mathcal{R}_0^2 + kt_0(t_0 - t))^2}, \tag{77}$$

$$\mathcal{R}(t) = \sqrt{\frac{t}{t_0}}\sqrt{\mathcal{R}_0^2 + kt_0(t_0 - t)}, \tag{78}$$

where $\mathcal{R}_0$ is the scale parameter at a fixed time t_0 during radiation dominance. Evolving back the current energy density, which is within one order of magnitude of the critical density, shows that $\mathcal{R}_0 \gg kt_0$ during radiation dominance and curvature effects can be neglected. The energy density can then be estimated to relate time and temperature scales[15]:

$$\varrho = \frac{\pi^2}{30}g(T)T^4, \tag{79}$$

where $g(T)$ is the effective number of relativistic degrees of freedom in thermal equilibrium and varies by a factor of two for temperatures between 100 MeV and 1 TeV: If we assume standard model particle content plus an axidilaton we find in the high temperature regime

[14]The robustness of these scales against our ignorance of particle physics at very high energies is amazing: Eqs. (77,79) show that supersymmetry would divide these time scales only by a factor $\sqrt{2}$, and that one would need 10^4 additional relativistic degrees of freedom to invalidate the order of magnitude estimates!

[15]Without approximate flatness we would find a curvature term $-k\mathcal{R}^{-2}(P_iX^i)^2$ in the dispersion relation, and even the treatment of ideal gases would become very complicated.

$g(1\text{TeV}) = 105.75$ including the Higgs particle[16] and the top quark, or $g(1\text{TeV}) = 94.25$ without them. In the low temperature sector we find $g(1\text{GeV}) = g(100\text{MeV}) = 49.75$, where the light particles included are the electron, up and down quarks, three left–handed neutrinos, the photon, eight gluons, the axion and the dilaton.

Thermally produced dilatons are relativistic, and their equilibrium density at temperature T is:

$$\nu(T) = \frac{\zeta(3)}{\pi^2} T^3.$$

From this and eqs. (77–79) we find a mild increase of the number of dilatons per comoving volume $v = V\mathcal{R}^{-3}$ with temperature:

$$N_\phi = \frac{\zeta(3)}{\pi^2}\Big(\frac{45}{2g(T)}\Big)^{3/4}\Big(\frac{m_{Pl}}{\pi t_0}\Big)^{3/2}\mathcal{R}_0^3.$$

This can easily be understood: As temperature approaches mass thresholds the effective number of relativistic degrees of freedom decreases and particles annihilate into the remaining light degrees of freedom.

We learn from $g(T)$ that thermally created dilatons contributed about 2% to the energy density of the universe for 1 GeV$> T >$100 MeV, if the dilaton was still in thermal equilibrium. On the other hand, we have seen that the dilaton is extremely weakly coupled, and it may well happen that it decouples from the heat bath at a temperature T_{dec} above 1 GeV. Then the energy density of dilatons which were thermally produced at the temperature T_{dec} is still governed by the temperature T of the heat bath as long as the heat bath remains relativistic. This holds for any massless decoupled particle species and is a simple consequence of the fact that the energy density of thermally produced decoupled species evolves according to $\varrho_{dec} \sim \mathcal{R}^{-4} \sim T^4$, i.e. the decoupled species cools out exactly like the relativistic heat bath. There is a difference, of course: The number of effective relativistic degrees of freedom seen by the decoupled particles was $g(T_{dec}) > g(T)$, and if thermally produced dilatons decoupled at T_{dec} their contribution to the energy density of the universe was $g(T_{dec})^{-1} < 2\%$, while their number density was reduced by a factor $(g(T)/g(T_{dec}))^{3/4}$. After photon recombination the contribution of thermal dilatons to Ω becomes negligible since $\varrho_{\phi,th}$ still decays with $\mathcal{R}^{-4}$, while the energy density in the dust decays only with $\mathcal{R}^{-3}$. Therefore, with or without decoupling thermally produced dilatons make no relevant contribution to the present energy density of the universe.

How then do the worries arise that the coupling scale of the dilaton is constrained to values below 10^{12} GeV from the requirement $\Omega \leq 1$, similar to the decay constant of the axion? This follows from the corresponding analysis for the axion in [80, 1, 37, 95], if f_ϕ is supposed to determine the expectation value $\langle\phi^2\rangle$ of the dilaton at the onset of coherent oscillations. However, there is a caveat in this reasoning: It follows from the form of the instanton induced potential for the axion that its amplitude at the onset of oscillations is of the order f_a, since both a periodic and a non–periodic axion is at most $|\Delta a| \leq \pi f_a$ away from a local minimum of $V(a)$. On the other hand, such an estimate makes no sense for

[16] Right–handed neutrinos were excluded in the calculation of $g(T)$.

the dilaton since the low energy potential is not periodic and we have encoded any non–vanishing expectation value of the dilaton in our horizon in the gauge coupling. We also should not rely on dimensional arguments, since at the onset of oscillations there are two widely different mass scales which govern the dynamics of the dilaton: A very large decay constant and a very small mass.

As a consequence, we employ the relation (68)

$$m_\phi f_\phi \simeq \sqrt{6} m_a f_a$$

to estimate the contribution of dilaton oscillations to Ω. Coherent oscillations of the axion and the dilaton arise when the mass terms begin to dominate over the expansion terms in the equations of motion of scalars:

$$\ddot{\phi} + \frac{3}{2t}\dot{\phi} + m^2\phi = 0. \tag{80}$$

Classical trajectories of the axion and the dilaton satisfy this equation approximately, since due to the large decay constants the axion–dilaton coupling and the couplings to gauge fields provide negligible corrections to the linearized theory.

As long as mass terms can be neglected scalar fields approach stationary values with deviations fading with $t^{-1/2}$. On the other hand the field oscillates with frequency m if the mass term dominates, and this misalignment mechanism promotes light scalars to cold dark matter even though the temperature exceeds the mass. As a very heuristic argument to identify the transition region between the two regimes one may require smooth transition of $\dot{\phi}$. This gives for the transition time $2\tilde{t} = 3m^{-1}$. For times $t \gg m^{-1}$ the energy density $\varrho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2$ stored in the oscillations behaves exactly like pressureless matter under expansion: Dine and Fischler [37] pointed out that the constant mass solution to (80) is

$$\phi(t) = t^{-\frac{1}{4}}(A_+ J_{\frac{1}{4}}(mt) + A_- J_{-\frac{1}{4}}(mt)),$$

and the asymptotic expansion for $mt \gg 1$ implies $\varrho_\phi \sim \mathcal{R}^{-3}$. This scaling behavior of ϱ_ϕ persists in a dust dominated universe, where ϕ evolves with $t^{-1/2}J_{\pm\frac{1}{2}}(mt)$ and $\mathcal{R}$ evolves with $t^{2/3}$, and the oscillations do not contribute to the pressure in the universe. Several other groups have analyzed the influence of the instanton induced axion mass and found that it begins to dominate over the expansion term at $T_a \simeq 1\text{GeV}$ [80, 1, 95].

A calculation of the temperature dependence of the axion mass by Gross, Pisarski and Yaffe was employed by Turner to determine the mass dependence of the temperature T_a beyond which the evolution of axions is dominated by oscillatory behavior [95]. He found that the transition temperature scales with m_a according to $T_a \sim m_a^{0.18}$, where m_a is the low temperature limit of the axion mass. Due to the constancy of the decay constants Turner's result also applies to the dilaton and we find for the corresponding scale

$$\frac{T_\phi}{T_a} \simeq \left(\frac{f_a}{f_\phi}\right)^{0.18},$$

whence the momenta of the oscillations entering the horizon are related by

$$\frac{p_\phi}{p_a} \simeq \frac{t_a}{t_\phi} \simeq \left(\frac{f_a}{f_\phi}\right)^{0.36}.$$

From these ratios follows an estimate on the velocity ratio, which can be used as a further indicator that coherent dilaton oscillations qualify as cold dark matter:

$$\frac{v_\phi}{v_a} \simeq \left(\frac{f_\phi}{f_a}\right)^{0.64}.$$

To discuss implications of the previous results on the role of the dilaton in cosmology we should distinguish two cases:
– S–duality applies at the GUT scale, implying that f_ϕ is bounded to be at most two orders of magnitude above the expected value for a misaligned axion $f_a \simeq 10^{12}$ GeV.
– S–duality is broken, with $f_a \simeq 10^{12}$ GeV but $f_\phi \simeq 10^{18}$ GeV.

The consequences in the first case are schematically similar to the consequences in the second case, but it leaves us with the puzzle to identify a mechanism which could lower f_ϕ by four orders of magnitude from its theoretically expected value. A further case very similar to the case of S–duality at the GUT scale would suppose S–duality at the QCD scale. A priori there seems no particular justification for this assumption, apart from the fact that it nicely complies with the mass estimate $m_\phi \geq 10^{-4}$ eV which would arise for a dilaton coupling to nucleon masses [43]. Then the dilaton decay constant would be close to the axion decay constant and the axion and the dilaton would develop oscillations at the same scale and make comparable contributions to Ω.

In the second case S–duality is maintained in the axidilaton sector, but not in the couplings to gauge fields. Eq. (68) then hints at a non–perturbatively generated dilaton mass which is much smaller than the axion mass:

$$m_\phi \sim 10^{-6} m_a$$

and the dilaton will start to oscillate after the axion, when the temperature has dropped by another factor of 10 and the time scale has expanded by two orders of magnitude.

The velocity of large scale fluctuations entering the horizon at the QCD scale is $v_\phi \sim 10^4 v_a$, and from $v_a \sim 10^{-6}$ [15] we learn that even in this sense dilaton oscillations remain non–relativistic for all choices of f_ϕ. Borrowing on the results of [80, 1, 37, 95] for the axion we find for the dilaton contribution to the contemporary energy density of the universe

$$\frac{\Omega_\phi}{\Omega_a} \simeq 10^{-5} \frac{\langle\phi^2\rangle}{f_a^2} \simeq 10^7 \frac{\langle\phi^2\rangle}{m_{Pl}^2}, \tag{81}$$

and the dilaton would make an appreciable contribution to the energy density for misalignment in the range

$$\frac{\sqrt{\langle\phi^2\rangle}}{m_{Pl}} \sim 10^{-3} - 10^{-4}.$$

Taking into account the time evolution of the dilaton before instanton tunneling this corresponds to

$$\frac{\sqrt{\langle \phi^2 \rangle}}{m_{Pl}} \simeq 1$$

for temperatures above 1 TeV, and the outlook for an appreciable fraction of dilatons in the dark matter seems promising. However, any further investigation of this subject requires better knowledge, or speculation, about new physics and evolution of a massless dilaton for temperatures above 1 TeV, and this is beyond the scope of the present work.

9 Conclusions and outlook

The appearance of fundamental scalars with a direct coupling to gauge curvature terms remains a challenge in string theory which offers unexpected rewards in low energy physics.

In order to resolve the ambiguity in the definition of gauge couplings in the presence of a massless dilaton, the dilaton has to acquire a mass at an early stage in the evolution of the universe. Motivated from the observation that both in string theory and in Kaluza–Klein theory the dilaton couples with different signs to axions and to gluons the proposal was made that rather than a gluino condensate it is a variance of the axion in the Euclidean domain which stabilizes the dilaton. For consistency this proposal has to rely on the assumption that the four–dimensional field theory containing axions and dilatons is an effective theory, with topological defects stabilized through higher derivative terms, as is the case e.g. in string theory. An S–dual coupling between the axion and the dilaton then yields an estimate on the dilaton mass $m_\phi \simeq m_a f_a {f_\phi}^{-1}$. Comparison with the simplest supersymmetric extension of an axidilaton–gluon theory revealed that the axion coupling should dominate the dilaton mass for decay constants $f_\phi > 10^{11}$ GeV and gluino condensates below 1 TeV^3.

We have pointed out that the dynamics of a light scalar in an expanding universe before mass dominance easily accomodates for large coupling scales without overclosing the universe as long as no multivalued vacua emerge. For a dilaton with coupling scale $f_\phi \simeq m_{Pl}$ this means that a variance $\sqrt{\langle\phi^2\rangle} \simeq m_{Pl}$ is still permissible at a temperature $\simeq 10^3 T_\phi$, where T_ϕ is the temperature where coherent dilaton oscillations evolve. In this case we have seen that the dilaton provides an interesting candidate for cold dark matter accompanying an axionic component, and for a coupling to QCD we found an estimate $T_\phi \simeq 10^{-1} T_a$. As a consequence the onset of dilaton oscillations seems to be close to or even coincide with the QCD phase transition.

We have also seen that a dilaton coupling to gauge curvature terms provides a simple mechanism to accomodate both a regularized Coulomb potential and a confining potential in gauge theory. Given this observation and the fact that string theory unavoidably predicts a dilaton coupling to gauge fields, continuing investigation of light dilatonic degrees of freedom seems more than justified. It is of particular interest to see how the transition from the confining solution to a regularized Coulomb potential proceeds, and which parameters control the phases of a gauge theory coupling to a dilaton.

In conclusion, gauge theories with a dilaton present rather an interesting than a worrisome prediction of string theory.

Appendix A: Conventions and notation

We use Greek letters for three– and four–dimensional holonomic indices, while anholonomic indices are denoted by Latin letters from the beginning of the alphabet. Higher dimensional holonomic tangent frame indices are denoted by capital letters from the middle of the alphabet and anholonomic indices by capital letters from the beginning of the alphabet. Hence, components of the 4–bein and the D–bein in $D > 4$ dimensions read $e_\mu{}^a$, $E_M{}^A$. Latin letters from the middle of the alphabet are used both for Lie algebra indices and for spatial Minkowski space indices in $3 + 1$ dimensions, and matrix elements of Lie algebra generators are written as $(X_i)_{ab}$. We use a boldface notation for 3–vectors. Gauge couplings are usually denoted by q, while g is reserved for the metric in three or four dimensions.

Our conventions for Planck units are rescaled by a factor $\sqrt{8\pi}$:

$$m_{Pl} = \kappa^{-1/2} = (8\pi G)^{-1/2} = 2.4 \times 10^{18}\,\mathrm{GeV},$$

$$t_{Pl} = 2.7 \times 10^{-43}\,\mathrm{s},$$

$$l_{Pl} = 8.1 \times 10^{-35}\,\mathrm{m}.$$

In the literature these units are sometimes referred to as reduced Planck units.

The generic setting for quantum field theory are total spaces fibered by a usually highly reducible representation space of a group $SO(1, D-1) \times G$. G is referred to as a gauge group, and is assumed to be a compact Lie group consisting of simple factors. It is generated by a Lie algebra with relations

$$[X_i, X_j] = i f_{ij}{}^k X_k.$$

The fiber projects down to a D–dimensional base space $\mathcal{M}$ of Minkowski signature $(-,+,\ldots,+)$, and $SO(1,D-1)$ is the structure group of the tangent bundle. The generators of $SO(1, D-1)$ as well as their representations are denote by $L_{ab} = -L_{ba}$.

Covariant derivatives and curvatures are defined via

$$D_\mu = \partial_\mu + \omega_\mu - iqA_\mu = \partial_\mu - \frac{1}{2}\omega^a{}_{b\mu} L^b{}_a - iqA_\mu{}^j X_j,$$

$$R_{\mu\nu} - iqF_{\mu\nu} = -\frac{1}{2} R^a{}_{b\mu\nu} L^b{}_a - iqF_{\mu\nu}{}^j X_j = [D_\mu, D_\nu],$$

and the dual curvature tensor $\tilde{F}$ in four dimensions is

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}.$$

F is denoted as self–dual in Minkowski space if

$$\tilde{F}_{\mu\nu} = iF_{\mu\nu}.$$

A Wick rotation maps the Schrödinger equation into the diffusion equation through[17]

$$t \to -i\tau$$

and the transition from Minkowski space field theory to Euclidean field theory proceeds via

$$\varphi(t, \mathbf{x}) \to i^{\#(0)} \varphi_E(\tau, \mathbf{x}),$$
$$\mathcal{L}(\varphi) \to -\mathcal{L}_E(i^{\#(0)} \varphi_E),$$

where $\#(0)$ denotes the net number of covariant timelike indices in the field φ. The ϵ–tensor is not covariantly transformed under Wick rotation and satisfies $\epsilon^{0123} = \frac{1}{\sqrt{|g|}}$, thus accounting for the oscillatory instanton contribution through the axion–gluon coupling.

On the level of partition functions

$$Z[J] = \exp(iW[J]) = \int D\varphi \exp(iS[\varphi] + i \int d^4x J \cdot \varphi)$$

is mapped to

$$Z_E[J_E] = \exp(-W_E[J_E]) = \int D\varphi_E \exp(-S_E[\varphi_E] + \int d^4x J_E \cdot \varphi_E).$$

The mean fields are

$$\phi(x) = \frac{\delta W[J]}{\delta J(x)},$$
$$\phi_E(x) = -\frac{\delta W_E[J_E]}{\delta J_E(x)},$$

and the effective actions are accordingly

$$\Gamma[\phi] = W[J] - \int d^4x J(x) \cdot \phi(x),$$

$$\Gamma_E[\phi_E] = W_E[J_E] + \int d^4x J_E(x) \cdot \phi_E(x).$$

The effective actions and the mean fields thus encode the quantum dynamics of the system under consideration in terms of classical evolution equations:

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -J(x),$$
$$\frac{\delta \Gamma_E[\phi_E]}{\delta \phi_E(x)} = J_E(x).$$

Note that under Wick rotation $\delta(x) \to i\delta(x)$, and therefore $J(t, \mathbf{x}) \to (-i)^{\#(0)} J_E(\tau, \mathbf{x})$, but

$$\frac{\delta}{\delta J(x)} \to i^{\#(0)+1} \frac{\delta}{\delta J_E(x)}.$$

The mean fields thus transform like the quantum fields: $\phi(x) \to i^{\#(0)} \phi_E(x)$.

[17] van Nieuwenhuizen and Waldron recently pointed out that the Wick rotation has a continuous extension in terms of a five–dimensional Lorentz–boost. They employed this observation to identify the action of the Wick rotation on spinors [74].

Appendix B: A remark on perturbative aspects of the axidilaton

We have argued from the Kaluza–Klein type coupling of a dilaton to gauge fields that instantons should provide a mechanism to stabilize a dilaton in four dimensions, irrespective from the presence or absence of higher massive modes. It is tempting to push this argument a little further and conclude that no effective potential should be generated perturbatively: In a low–dimensional Kaluza–Klein theory the volume of internal dimensions effectively only rescales the higher–dimensional Planck mass and axion constant to their low–dimensional values, and we have seen that the axion and dilaton couplings carry no further remembrance of the compactification scale. Thus in the low energy approximation the compactification scale reduces to a mass threshold, but there is no traceable imprint of this scale in the low energy sector[18]. Hence perturbation theory in the low energy sector can not reveal the presence of a compactification scale or its actual value, and therefore it can not remove the degeneracy of the dilaton. The same conclusion then should apply to any theory with a dilaton as long as only Kaluza–Klein type couplings are considered. The shaky point about this argument concerns the non–renormalizability of the model under discussion and the question of the very meaning of perturbation theory. One is on much safer ground if supersymmetry can be employed to exclude a perturbatively generated dilaton potential [39], but going below the SUSY scale we have left that safe harbor behind. Nevertheless, it turns out that the reasoning is not in contradiction with a 1–loop calculation as long as one relies on dimensional regularization:

On the 1–loop level the effective potential is generated by 1–loop diagrams with only axions and dilatons of vanishing momenta as external particles. There appear three types of relevant tree level vertices in (52):
— For external axions with gauge bosons in the loop there is only one relevant vertex:

$$i\delta_{jk}\frac{q^2}{8\pi^2 f_a}\epsilon_{\mu\nu\rho\sigma}p_1^\rho p_2^\sigma, \tag{82}$$

where the gauge bosons at the vertex have momenta p_1, p_2, polarizations μ, ν and orientations j, k in the Lie algebra. We have also taken out one factor of i into the momentum conserving factor $(2\pi)^4 i\delta(p_1+p_2+k)$, with k denoting the 4–momentum of the axion.
— For external dilatons with gauge bosons in the loop there are enumerably many vertices:

$$\delta_{jk}\frac{1}{(if_\phi)^n}(p_{1\nu}p_{2\mu}-\eta_{\mu\nu}p_1\cdot p_2). \tag{83}$$

— The corresponding axion–dilaton vertices are

$$-\left(\frac{2i}{f_\phi}\right)^n p_1\cdot p_2, \tag{84}$$

[18] Of course, in a real Kaluza–Klein system there would be scalars in the adjoint representation of the gauge group, and with chiral fermions one could infer the likely existence of a compactification scale, but one would not have any hint for its order of magnitude.

where now p_1 and p_2 are the incoming axion momenta.

From the vertices it is immediately clear that no perturbatively generated axion potential appears in the theory: (84) vanishes due to the vanishing axion momenta, while (82) vanishes due to momentum conservation with vanishing external axion momentum.

On the other hand, a diagram with external zero–momentum dilatons and axions or gluons in the loop is directly proportional to $\int d^4p$, and this vanishes in dimensional regularization [100]. After Wick rotation one finds

$$\int d^{4-2\varepsilon}p = \lim_{\alpha\to 0}\int d^{4-2\varepsilon}p\,\frac{1}{(p^2+m^2)^\alpha} = \lim_{\alpha\to 0} i\pi^{2-\varepsilon}m^{4-2\varepsilon-2\alpha}\frac{\Gamma(\alpha-2-\varepsilon)}{\Gamma(\alpha)} = 0$$

and no ϕ^n–vertices could be inferred from these diagrams.

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Charged Dilatonic Black Holes: String frame vs Einstein frame

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The descriptions of Reissner-Nordström and Kerr-Newman dilatonic black holes in the Einstein frame are compared to those in the string frame. We describe various physical measurements in the two frames and show which experiments can distinguish between the two frames. In particular we discuss the gyromagnetic ratios of black holes, the decay law via Hawking radiation and the propagation of light on black hole backgrounds.

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I. INTRODUCTION

Superstring theory [1] compactified down to physical four space-time dimensions is generally accepted as the description of curved backgrounds as non-vanishing expectation values of massless string excitations in a language which reproduces Einstein's general relativity. However it is not *a priori* clear in which frame such a description should be better formulated, since there are several frameworks possible.

Specifically, upon compactification of extra dimensions and nearly independently of both the content of the starting theory and the compactification scheme, one expects to obtain a low-energy, tree-level, effective 4-dimensional action S_σ for the evolution of the uncompactified degrees of freedom of the string, which is in the form of a conformally invariant non-linear σ-model. A spin 2 gravitational field G_{ij} (Latin indices run from 0 to 3) and an antisymmetric field B_{ij} which couple respectively to the string vertex operators for their own emission (see [2,3] and Refs. therein), emerge in a natural way in S_σ

$$S_\sigma = \frac{1}{2\,\lambda_s^2}\,\int d^4x\,\left[h^{\alpha\beta}\,G_{ij}\,\partial_\alpha x^i\,\partial_\beta x^j + \epsilon^{\alpha\beta}\,B_{ij}\,\partial_\alpha x^i\,\partial_\beta x^j + \ldots\right]\ , \tag{1.1}$$

where λ_s is the string length, $h_{\alpha\beta}$ the world-sheet metric tensor, $\epsilon^{\alpha\beta}$ the Levi-Civita symbol in two dimensions and ellipses stand for fermionic terms as well as gauge fields and other moduli.

Of course were one able to perform the derivation completely, the background expectation values of the various fields could be fully determined from the action of the fundamental theory from which one has started. This purely descriptive picture is hard to perform in general, if possible at all. The most serious obstacle to obtaining such a picture is that there are at present no models which describe compactification as a dynamical process taking place in the higher dimensional space of (super)string theory and leading to an evolution towards the present state of the Universe as the preferred vacuum (for a review of early attempts see *e.g.* [4]). Thus one is naturally led to seek other ways of solving for the fields. A possible guide is the requirement of conformal invariance on the world-sheet for a generalized D-dimensional σ-model which furnishes us a complete set of renormalization equations. Further, it turns out that in the latter set one has also to include the scalar excitation ϕ (the dilaton) which couples to the world-sheet scalar curvature [3].

The aforementioned equations, supplemented with suitable boundary conditions, are sufficient to determine the background fields without making any further reference to string theory. This is a result of the fact that the same equations can also be obtained by varying an effective action which, for the particular case of $B_{ij} \equiv 0$ and non zero Maxwell strength tensor of electrodynamics F_{ij}, can be written (see [5], but we work in four space-time dimensions)

$$S_{SF} = \frac{1}{2}\,\int d^4x\,\sqrt{-G}\,e^{-\phi}\,\left[\frac{1}{\lambda_s^2}\,\left(R_{(G)} + G^{ij}\,\nabla_i\phi\,\nabla_j\phi\right) - \frac{1}{\alpha^2}\,e^{(1-a)\,\phi}\,F^2\right]\ , \tag{1.2}$$

where $R_{(G)}$ is the scalar curvature of the metric G_{ij}, α is the electromagnetic coupling constant and a is the dilaton coupling constant with the dilaton assumed to remain massless [6]. This is the effective action which describes the

background fields of the selected string vacuum in the so called *string frame* (SF). The name is justified by the fact that the uncompactified degrees of freedom of the string move along a geodesic of the metric G_{ij} as can be inferred from (1.1). The equations of motion following from (1.2) can be written as (we omit the subscript (G) and set $\lambda_s = \alpha = 1$)

$$\begin{aligned} & R_{ij} - \frac{1}{2}\, G_{ij}\, R + \frac{1}{2}\, G_{ij}\, (\nabla\phi)^2 - G_{ij}\, \nabla^2\phi + \nabla_i\nabla_j\phi - 2\, e^{(1-a)\,\phi}\, T^{EM}_{ij} = 0 \\ & \nabla^2\phi - (\nabla\phi)^2 + a\, e^{(1-a)\,\phi}\, F^2 = 0 \\ & \nabla_i \left(e^{-a\,\phi}\, F^{ij}\right) = 0 \ , \end{aligned} \tag{1.3}$$

where ∇ denotes the covariant derivative with respect to the metric G_{ij} and the electromagnetic energy-momentum tensor is given as

$$T^{EM}_{ij} = F_{ik}\, F^k_j - \frac{1}{4}\, G_{ij}\, F^2 \ . \tag{1.4}$$

The effective action and equations of motion can be further modified by applying the following conformal transformation

$$G_{ij} = e^{\phi-\phi_0}\, g_{ij} \ , \tag{1.5}$$

where the constant ϕ_0 can be taken to be the average value of the dilaton in the present Universe. Since in these notes we consider only asymptotically flat cases, we will assume $\phi_0 \equiv 0$ in the forthcoming sections. We further define the Planck length as

$$\ell_p^2 = e^{\phi_0}\, \lambda_s^2 \ , \tag{1.6}$$

thus obtaining the action in the *Einstein frame* (EF),

$$S_{EF} = \frac{1}{2} \int d^4x\, \sqrt{-g}\, \left[\frac{1}{\ell_p^2} \left(R_{(g)} - \frac{1}{2}\, g^{ij}\, \nabla_i\phi\, \nabla_j\phi \right) - \frac{1}{\alpha^2}\, e^{-a\,\phi}\, F^2 \right] \ , \tag{1.7}$$

in which we point out that space-time coordinates have been left unchanged and $R_{(g)}$ is the curvature related to g_{ij}. The new equations of motion are obtained by applying the same mapping to the previous ones given in (1.3) (again we omit the subscript (g) and set $\ell_p = \alpha = 1$),

$$\begin{aligned} & R_{ij} = \frac{1}{2}\, \nabla_i\phi\, \nabla_j\phi + 2\, e^{-a\,\phi}\, T^{EM}_{ij} \\ & \nabla^2\phi + a\, e^{-a\,\phi}\, F^2 = 0 \\ & \nabla_i(e^{-a\,\phi}\, F^{ij}) = 0 \ , \end{aligned} \tag{1.8}$$

where ∇ is now the covariant derivative with respect to the metric g_{ij}. We observe that the dilaton is obviously left unchanged and that the equation for the electromagnetic field is (formally) the same as in SF. Indeed, since $\nabla_i(e^{-a\,\phi}\, F^{ij}) \equiv \partial_i(\sqrt{-g}\, e^{-a\,\phi}\, [F^{ij}]_{EF}) = \partial_i(\sqrt{-G}\, e^{-a\,\phi}\, [F^{ij}]_{SF})$ from (1.3) and (1.8), solutions of Maxwell's equations in the two frames must be related by

$$e^{2\,\phi}\ \left[F^{ij}\right]_{SF} = \left[F^{ij}\right]_{EF} \ . \tag{1.9}$$

Therefore the physical (covariant) components of the electromagnetic field are the same in both frames:

$$[F_{ij}]_{SF} = [F_{ij}]_{EF} \ . \tag{1.10}$$

Furthermore, the uncompactified degrees of freedom of the string do not move along a geodesic of the metric g_{ij}, and the scalar curvatures are in general different in the two frames because of the dilaton,

$$\begin{aligned} & R_{(G)} = 2\, (\nabla\phi)^2 - 3\, \nabla^2\phi \\ & R_{(g)} = \frac{1}{2}\, (\nabla\phi)^2 \ . \end{aligned} \tag{1.11}$$

Another general observation is that S_{EF} is invariant under the following transformation [7]

$$\mathcal{T} : \begin{cases} a \to -a \\ \phi \to -\phi \end{cases} , \tag{1.12}$$

which is not an invariance of S_{SF}. Therefore, although the mapping (1.5) looks almost trivial at a first sight, it is clear that the physics in the two frames can be significantly different.

Which frame is more suitable as a description of the present state of our Universe is an open question which will eventually be settled by experiment [8]. The issue raised in the present notes has already been extensively discussed in the framework of scalar-tensor theories of gravity and observable consequences have been deduced mainly in cosmology. Because of the direct coupling between the dilaton and matter (in our case the electromagnetic field), both actions in Eqs. (1.2) and (1.7) fail to be of the Brans-Dicke type, thus the equivalence principle does not hold in general. Specifically, one expects the equivalence principle to be violated whenever the gradient of the dilaton field is not negligible. However, such violations might be allowable provided they occurred far in the past, *e.g.* in the early stages of the Universe (see [4], [9] and Refs. therein, [10]) or take place in regions of space which have not been tested at present.

In the following we will analyze some of the physical implications of the differences between the two frames for two black hole geometries, in which the gradient of the static dilaton field is appreciably strong only in a relatively small region of space outside the event horizon. We consider a charged black hole (RND) [11,5] in section II and a rotating black hole with small charge-to-mass ratio (KND) [12] in section III and suggest possible experiments. Finally, in section IV we compare the propagation of light on these black hole backgrounds in the two frames.

II. RND BLACK HOLES

The line element representing a 4-dimensional charged dilatonic black hole in EF is [11,5]

$$ds^2\big]_{EF} = -e^{2\,\Phi}\,dt^2 + e^{2\,\Lambda}\,dr^2 + R^2\,d\Omega_2^2 \ , \tag{2.1}$$

with $d\Omega_2^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$ and

$$\begin{aligned} e^{2\,\Phi} &= e^{-2\,\Lambda} = \left(1 - \frac{r_+}{r}\right)\,\left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} \\ R^2 &= r^2\,\left(1 - \frac{r_-}{r}\right)^{\frac{2\,a^2}{1+a^2}} \ . \end{aligned} \tag{2.2}$$

One encounters an outer horizon at

$$r_+ = M + \sqrt{M^2 - (1-a^2)\,Q^2} \ , \tag{2.3}$$

while

$$r_- = (1+a^2)\,\frac{Q^2}{r_+} \tag{2.4}$$

is a real singularity for $a \neq 0$. The corresponding electromagnetic field has only one non-vanishing component (the static electric field),

$$F_{tr} = \frac{Q}{r^2} \ , \tag{2.5}$$

and the dilaton field is given by

$$e^{-\phi} = \left(1 - \frac{r_-}{r}\right)^{-\frac{2\,a}{1+a^2}} = 1 + \mathcal{O}(r^{-1}) \ . \tag{2.6}$$

We observe that for $a = 0$ the above expressions reduce to the pure Reissner-Nordström (RN) solution (with a constant $\phi_0 = 0$ dilaton field) and that M and Q represent the physical (ADM) mass and charge of the black hole.

As mentioned in the Introduction, from (2.6) one sees that the dilaton gradient falls off and becomes negligible sufficiently far away from r_-. To be more specific, since $e^{\phi} \sim G_N$, one can write the total force acting on a test mass m as the sum of the force F_ϕ due to the spatial dependence of G_N and the Newtonian contribution F_N,

$$F_{tot} \sim -\partial_r \left(G_N \, \frac{M\, m}{r} \right) = F_\phi + G_N \, \frac{M\, m}{r^2} \ , \tag{2.7}$$

and finds that F_ϕ becomes of the same order as F_N at

$$r \sim \frac{1+3\,a^2}{1+a^2}\, r_- \ . \tag{2.8}$$

This implies that, if one can perform a measurement with the precision of one part over 10^N, one has to go closer than $r_c \sim 10^N\, r_-$ to the black hole centre in order to test any violation of the equivalence principle. Further, r_c must lie outside r_+ and this gives an estimate for the smallest charge-to-mass ratio that the black hole must posses in order that any deviation can be tested, namely

$$\frac{Q}{M} > 10^{-N/2} \ . \tag{2.9}$$

For a solar mass black hole and $N \sim 10$ this means a charge of about 10^{34} electron charges or 10^{15} C. On the other hand, for a Planck mass black hole with one electron charge the ratio $Q/M \sim 0.1$ and one only needs a precision of $N = 2$.

Upon transforming to SF one obtains

$$ds^2]_{SF} = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1+2\,a-a^2}{1+a^2}}\, dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{\frac{a^2+2\,a-1}{1+a^2}}\, dr^2 + r^2\left(1-\frac{r_-}{r}\right)^{\frac{2\,a\,(1+a)}{1+a^2}}\, d\Omega_2^2 \ . \tag{2.10}$$

An interesting feature of the latter metric is that $G_{\theta\theta}$ (as well as $G_{\varphi\varphi}$) is regular at $r = r_-$ for $a = 0$ (RN), as expected, and also for $a = -1$ [13]. Further, the physical (ADM) mass of the black hole is shifted according to

$$M_{phys}]_{SF} = M + \frac{a\,Q^2}{r_+} = M\left(1 + \frac{a\,Q^2}{2\,M^2}\right) + \mathcal{O}\left(\frac{Q^4}{M^4}\right) \ . \tag{2.11}$$

The latter result is however not particularly interesting from the experimental point of view unless a way can be found to measure M and M_{phys} *separately*.

As models of astrophysical black holes, both the line elements (2.1) and (2.10) are unsatisfactory. They describe non-rotating spherical objects, but real black holes are expected to be spinning fast because of angular momentum conservation during the collapse of the original star (for further evidence see *e.g.* [14]). If this limitation is ignored, the fact that M also appears in the expression for r_+ (and r_- as well, but the latter singularity is hidden) allows us to propose an experimental test consisting of the following steps:
i) by comparing the acceleration of a charged test particle to the acceleration of a neutral particle of equal mass the electric field F_{tr} can be measured;
ii) the value of M_{phys} is obtained directly from the acceleration of the neutral particle at large distance;
iii) the radius r_+ can be estimated by inferring the largest distance from which light can escape or by localizing the inner edge of the accreting disk.
F_{tr} is still given by Eq. (2.5), as can be inferred from the general relation (1.10). Thus step *i)* allows the computation of Q and the insertion of Q into the definition of r_+ in (2.3) which, together with the measured value of r_+ from *iii)*, gives M. If M is equal to M_{phys} from *ii)*, then EF is the physical frame and one might question the stringy origin of the action S_g; in case they are not equal, SF is the physical frame and (2.11) can be used to estimate a [6].

As a further consequence of the metric being different in the two frames, we notice that the expression of the area of the surface of the outer horizon is given respectively by

$$\begin{aligned} \mathcal{A}]_{EF} &= 4\,\pi\, r_+^{\frac{2}{1+a^2}}\ (r_+ - r_-)^{\frac{2\,a^2}{1+a^2}} \\ \mathcal{A}]_{SF} &= 4\,\pi\, r_+^{\frac{2\,(1-a)}{1+a^2}}\ (r_+ - r_-)^{\frac{2\,a\,(1+a)}{1+a^2}} \ , \end{aligned} \tag{2.12}$$

so that

$$\frac{\mathcal{A}]_{EF}}{\mathcal{A}]_{SF}} = \left(\frac{r_+ - r_-}{r_+}\right)^{\frac{2\,a}{1+a^2}} < 1 \ , \tag{2.13}$$

for $a > 0$ (the same ratio is > 1 for $a < 0$). As we shall show, this affects the evaporation of the black hole.

We assume that the area law holds in both frames, so that the internal degeneracy of the black hole is given by (see [15] and Refs. therein)

$$\Omega \sim e^{\mathcal{A}/4} \ , \tag{2.14}$$

where $\mathcal{A}$ is the area of the horizon. Then we study the evolution of the mass of the black hole in time assuming it can only decrease by emitting Hawking quanta (matter possibly falling into the black hole and increasing its mass will not be considered). The occupation number density of the Hawking radiation, properly computed in the microcanonical ensemble where total energy of the system is conserved and equal to M_{phys}, is [15]

$$n(\omega) = \sum_{l=1}^{M_{phys}/\omega} \frac{\Omega(M_{phys} - l\,\omega)}{\Omega(M_{phys})} \ . \tag{2.15}$$

For the sake of simplicity we specialize to the case $a = 1$ and also assume that the ratio $x \equiv Q/M$ is small and remains constant along the evaporation. Thus we find for the area of the horizon

$$\mathcal{A} \simeq 4\,\pi\,M^2\,\left(1 - f\,x^2\right) \ , \tag{2.16}$$

where M is now the physical mass as measured in each frame and $f = 1/2$ in EF ($f = 3/2$ in SF). The occupation number (2.15) becomes

$$n_f(\omega) \sim \sum_{l=1}^{M/\omega} \left[e^{4\,\pi\,l^2\,\omega^2 - 8\,\pi\,M\,l\,\omega}\right]^{(1-f\,x^2)} \ . \tag{2.17}$$

from which the energy emitted per unit time can be computed according to

$$\frac{dM}{d\tau} \sim -\mathcal{A} \int d\omega\,\omega^3\,\Gamma(\omega)\,n_f(\omega) \ . \tag{2.18}$$

We have numerically integrated the above expression and the results for the two frames are shown in Fig. 1 for $\Gamma = 1$ and $x = 1/2$ (a relatively large ratio chosen for the purpose of stressing the difference between the two frames). From that figure one sees that for large values of M the emission is higher in SF, thus leading to a faster decay, but then the curve in EF overcomes the curve in SF for values around the Planck mass $M_p \equiv \sqrt{\hbar\,c/G_N}$ and smaller. Both curves reach a maximum and then vanish for zero mass, a feature which is a direct consequence of the use of the microcanonical approach (that is energy conservation).

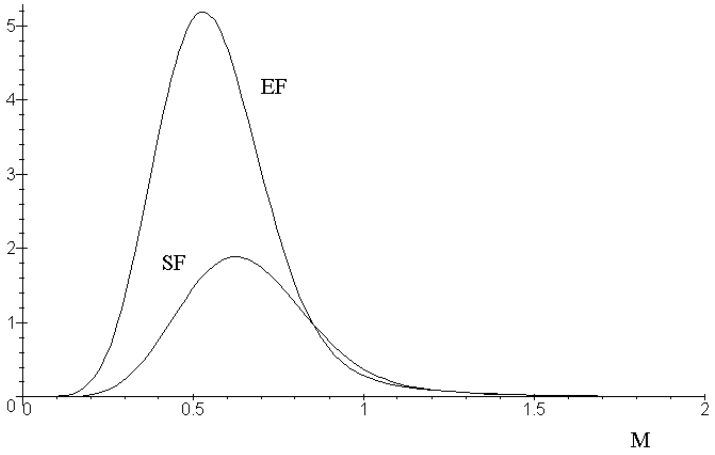

FIG. 1. Energy emitted by RND black holes per unit time and ratio Q/M fixed in the two frames. The mass is in units of the Planck mass. The vertical scale is arbitrary.

The time evolutions which result from Eq. (2.18) are shown in Fig. 2. When the mass M is of the order of the Planck mass (or bigger), Eq. (2.18) can be approximated by the thermal distribution with inverse temperature $\beta = 8\,\pi\,M\,(1 - f\,x^2)$. This behaviour would lead to a complete evaporation in a finite time. However, as can be seen from Fig. 2, after the maximum slope is reached the curves switch to a power law decay and fall to zero in infinite time (see Ref. [15] for the details). The difference between the two frames is that M is smaller in SF than in EF during the thermal phase but falls off less rapidly in SF than in EF during the late stages.

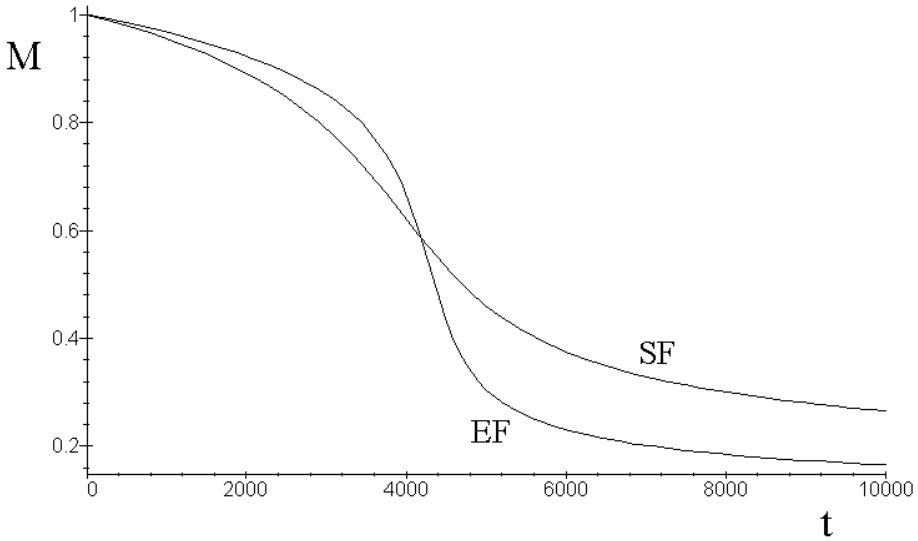

FIG. 2. Time evolution of the mass of RND black holes with $M(0) = M_p$ in the two frames. The time scale is arbitrary.

III. KND BLACK HOLES

In Ref. [12] a new solution of the Einstein-Maxwell dilatonic equations (1.8) was found in EF which represents a rotating, charged black hole with static dilaton field in the small charge-to-mass ($Q/M \ll 1$) approximation. In a subsequent paper [16] we showed that the corresponding metric can be simplified by shifting the radial coordinate r and suitably redefining the parameters M (mass), J (angular momentum) and Q (charge) to make them equal the corresponding physical (ADM) quantities M_{phys}, J_{phys} and Q_{phys} which determine the Newtonian motion in the asymptotically flat region. One then has

$$g_{ij} = g^{KN}_{ij} + \mathcal{O}\left(\frac{Q^4}{M^4}\right)\ , \tag{3.1}$$

where g^{KN}_{ij} is the usual Kerr-Newman (KN) metric which we write in Boyer-Lindquist coordinates [17],

$$ds^2_{KN} = -\sqrt{\Delta}\,\sin\theta\,\left[\chi\,d\varphi^2 - \frac{1}{\chi}\,(dt - \omega\,d\varphi)^2\right] + \rho^2\,\left[\frac{(dr)^2}{\Delta} + (d\theta)^2\right]\ , \tag{3.2}$$

in which ($\alpha \equiv J/M$)

$$\begin{aligned}
\chi &= \frac{\sqrt{\Delta}\,\sin\theta}{\Psi} \\
\Delta &= r^2 - 2\,M\,r + \alpha^2 + Q^2 \\
\rho^2 &= r^2 + \alpha^2\,\cos^2\theta \\
\Psi &= -\frac{\Delta - \alpha^2\,\sin^2\theta}{\rho^2} \\
\omega &= -\alpha\,\sin^2\theta\,[1 + \Psi^{-1}]\ .
\end{aligned} \tag{3.3}$$

This also implies that the causal structure is not affected by the presence of the dilaton at that order in EF. In fact one still has two horizons for $\Delta = 0$, that is

$$r_\pm = M \pm \sqrt{M^2 - \alpha^2 - Q^2} \ . \tag{3.4}$$

However the presence of a non-zero dilaton field,

$$\phi = -a \, \frac{r}{\rho^2} \, \frac{Q^2}{M} \ , \tag{3.5}$$

affects the electric and magnetic field potentials A and B (see Ref. [12] for the definitions),

$$\begin{aligned} A &= Q \, \frac{r}{\rho^2} \, \left[1 - \left(\frac{1}{2\,r} + \frac{r}{\rho^2} \right) \, \frac{a^2 \, Q^2}{3 \, M} \right] \\ B &= -Q \, \alpha \, \frac{\cos\theta}{\rho^2} \, \left[1 - \left(\frac{1}{2\,M} - \frac{r}{\rho^2} \right) \, \frac{a^2 \, Q^2}{3 \, M} \right] \ , \end{aligned} \tag{3.6}$$

where the terms proportional to Q^2 inside the brackets correspond to the corrections with respect to the KN potentials [17]. These corrections also affect the electric and magnetic fields in the limit $r \to \infty$,

$$\begin{aligned} \mathcal{E}_{\hat{r}} &\approx \frac{Q}{r^2} \\ \mathcal{E}_{\hat{\theta}} &\approx -\frac{2\,\alpha^2 \, Q}{r^4} \, \sin\theta \, \cos\theta \\ \mathcal{B}_{\hat{r}} &\approx \frac{2\,\alpha \, Q}{r^3} \, \cos\theta \, \left[1 - \frac{a^2 \, Q^2}{6 \, M^2} \right] \\ \mathcal{B}_{\hat{\theta}} &\approx \frac{\alpha \, Q}{r^3} \, \sin\theta \left[1 - \frac{a^2 \, Q^2}{6 \, M^2} \right] \equiv \frac{\mu_{phys}}{r^3} \, \sin\theta \ , \end{aligned} \tag{3.7}$$

where $\mathcal{E}_{\hat{a}}$ and $\mathcal{B}_{\hat{a}}$ ($\hat{a} = \hat{r}, \hat{\theta}$) are respectively the electric and the magnetic field components with respect to the usual spherical coordinate tetrad basis [18]. Eq.(3.7) shows a relative shift in the intensity of the field $\mathcal{B}$ with respect to $\mathcal{E}$. This is also the source for the anomalous gyromagnetic ratio

$$[g]_{EF} = 2 \, \frac{\mu_{phys} \, M_{phys}}{Q_{phys} \, J_{phys}} \simeq 2 \, \left[1 - \frac{a^2 \, Q^2}{6 \, M^2} \right] \ , \tag{3.8}$$

which for the KND black hole cannot be greater than 2 and is equal to 2 for the KN black hole [19].

An obvious consequence of the above form of the solution is that the geodesic motion of neutral particles, which do not couple directly to the dilaton, are unaffected by the presence of static dilaton field (up to order Q^3/M^3). As we have mentioned in the Introduction, this should not be the case for fundamental strings, from whose action S_{SF} has been derived by compactifying extra dimensions. To see that this is indeed the case we compute the SF metric corresponding to g^{KN}_{ij} with the dilaton field given in (3.5), thus obtaining

$$G_{ij} = g^{KN}_{ij} \, \left(1 - a \, \frac{r}{\rho^2} \, \frac{Q^2}{M} \right) + \mathcal{O}\left(\frac{Q^4}{M^4} \right) \ . \tag{3.9}$$

Then we observe that (3.9) cannot be remapped into g^{KN}_{ij} by a change of coordinates since the curvature scalars are different in the two frames. In fact, from (1.11) one obtains

$$\begin{aligned} R_{(G)} &\approx 12 \, a \, \frac{Q^2}{M} \, \alpha^2 \, \cos 2\theta \, \frac{1}{r^3} \\ R_{(g)} &\approx \frac{a^2}{2} \, \frac{Q^4}{M^2} \, \frac{1}{r^4} \ , \end{aligned} \tag{3.10}$$

where, for the sake of simplicity, we have displayed only the behaviors for large r.

Now we analyze further physical differences between the two frames. To start with, since

$$G_{tt} \approx -\left[1 - \frac{2\,M}{r} \, \left(1 + \frac{a \, Q^2}{2 \, M^2} \right) \right] \ , \tag{3.11}$$

for $r \to +\infty$, the ADM mass in SF is shifted to

$$[M_{phys}]_{SF} = [M_{phys}]_{EF} \left(1 + \frac{a\,Q^2}{2\,M^2}\right) \ , \tag{3.12}$$

which is the same expression that was obtained on expanding the physical mass in Q/M for RND. Thus, the same experimental test described in Section II after Eq. (2.11) can be repeated for the present case. We remind the reader here that the KND metric is a more realistic candidate for the description of astrophysical black holes, since it describes black holes which possess angular momentum.

We notice that

$$G_{t\varphi} \approx -2\,\frac{\alpha\,M}{r}\,\sin^2\theta \equiv -2\,\frac{J_{phys}}{r}\,\sin^2\theta \ , \tag{3.13}$$

therefore J_{phys} is left unchanged by the conformal mapping as well as μ_{phys}, see (1.10) and (3.7). The gyromagnetic ratio in SF is then given by

$$[g]_{SF} \simeq 2\,\left[1 + \frac{a\,Q^2}{2\,M^2}\,\left(1 - \frac{a}{3}\right)\right] \ . \tag{3.14}$$

The above expression for $[g]_{SF}$ shows a remarkable difference with respect to that of EF, that is $[g]_{SF}$ can be either smaller than 2 (for $a < 0$ or $a > 3$), equal to 2 for $(a = 0, 3)$ or greater than 2 (for $0 < a < 3$). The existence of the latter case provides another way of testing which frame is the physical one. In fact, since $[g]_{EF}$ can be at most equal to 2, the measurement of a value greater than 2 for the gyromagnetic ratio of a black hole would prove that Physics has to be described in SF (this should indeed be the case [3]). On the other hand, the measurement of any value smaller than 2, although crucial for proving the existence of static dilaton field, would not suffice for discriminating between EF and SF, unless an independent way of measuring a along with the mass and charge of the black hole can be found.

IV. LIGHT PROPAGATION

In this section we study the differences which emerge in the propagation of electromagnetic signals in the two frames.

We start from the approximation of geometric optics and look at the deflection angle of a null ray impinging upon the black hole from far away and then escaping to $r = +\infty$. Since the relation between SF and EF is given by a conformal transformation of the metric, the naive expectation is that no changes occur. In fact it is easy to prove that the picture is exactly the same in both frames, and the eikonal path followed by any null ray is unaffected by the choice of the frame.

We consider for the metric in EF a generic form $g_{ij} = g_{ij}(r, \theta)$, with only one possible off-diagonal term ($g_{t\varphi}$), which can be easily specialized to both RND and KND. The Lagrangian for a null particle moving on such a metric and in the equatorial plane $\theta = \phi/2$ is given by [17]

$$2\,\mathcal{L} = g_{tt}\,\dot{t}^2 + 2\,g_{t\varphi}\,\dot{t}\,\dot{\varphi} + g_{\varphi\varphi}\,\dot{\varphi}^2 + g_{rr}\,\dot{r}^2 = 0 \ , \tag{4.1}$$

where a dot denotes the derivative with respect to an affine parameter λ. The conserved momenta are

$$\begin{aligned} p_t &= g_{tt}\,\dot{t} + g_{t\,\varphi}\,\dot{\varphi} \equiv E \\ p_\varphi &= g_{tt}\,\dot{\varphi} + g_{t\,\varphi}\,\dot{t} \equiv L \ , \end{aligned} \tag{4.2}$$

corresponding respectively to the energy and angular momentum of the null particle. In particular, from the conservation of p_φ one obtains

$$[\dot{\varphi}]_{EF} = D^{-1}\,(g_{tt}\,L - g_{t\varphi}\,E) \ , \tag{4.3}$$

with $D \equiv g_{tt}\,g_{\varphi\varphi} - g_{t\varphi}^2$, and, after substituting for $\dot{t}$ and $\dot{\varphi}$ in the Lagrangian,

$$\left[\dot{r}^2\right]_{EF} = \frac{D^{-1}}{g_{rr}}\,\left(g_{tt}\,L^2 + g_{\varphi\varphi}\,E^2 - 2\,g_{t\varphi}\,E\,L\right) \ . \tag{4.4}$$

From the above expressions for $\dot{\varphi}$ and $\dot{r}$ one finally obtains

$$\left[\frac{d\varphi}{dr}\right]_{EF} = \left[\frac{d\varphi}{d\lambda}\right]_{EF}\left[\frac{d\lambda}{dr}\right]_{EF} = \sqrt{\frac{g_{rr}}{D}}\,\frac{g_{tt}\,L - g_{t\varphi}\,E}{\sqrt{g_{tt}\,L^2 + g_{\varphi\varphi}\,E^2 - 2\,g_{t\varphi}\,E\,L}}\ . \tag{4.5}$$

The deflection angle is then given by

$$\Delta\varphi = 2\ (\varphi(\bar{r}) - \varphi(\infty)) - \pi\ , \tag{4.6}$$

where

$$\varphi(r) = \int^{r}\left[\frac{d\varphi}{dr}\right]_{EF} dr\ , \tag{4.7}$$

and $\bar{r}$ is the minimum radial coordinate reached by the particle, that is

$$[\dot{r}]_{EF}|_{\bar{r}} = 0\ . \tag{4.8}$$

Explicit expressions can then be obtained for the two kinds of black holes by simply substituting in the corresponding metric elements [20].

Switching to SF, one obtains

$$\begin{aligned}[\dot{r}]_{SF} &= e^{-\phi}\ [\dot{r}]_{EF}\\ [\dot{\varphi}]_{SF} &= e^{-\phi}\ [\dot{\varphi}]_{EF}\ .\end{aligned} \tag{4.9}$$

Thus, since $e^{-\phi}$ is a positive function, one finds that the turning point at which the particle stops approaching the black hole has radial coordinate $\bar{r}$ which does not depend on the frame. Further

$$\left[\frac{d\varphi}{dr}\right]_{SF} = \left[\frac{d\varphi}{dr}\right]_{EF}\ . \tag{4.10}$$

One can then conclude that $\Delta\varphi$ is frame-independent and that eikonal trajectories of null waves are exactly the same in both EF and SF. In particular, for KND this means that to lowest order in $(Q/M)^2$ eikonals do not sense the dilaton at all [20].

Of course the above conclusion does not preclude the possibility of detecting other differences in the propagation of light waves in SF with respect to EF. To begin with, we notice that, although the radial coordinate of the turning points defined by Eq. (4.8) is frame-independent, the actual proper distance to an observer placed at $r = r_o$ depends on the frame. If we consider the RND case and $a = 1$, we see that the difference between proper distances to r_o is given by

$$l]_{EF} = l]_{SF} + r_-\ \ln\left(\frac{r_o - r_+}{\bar{r} - r_+}\right)\ , \tag{4.11}$$

which would result in a delay for the time of flight from $\bar{r}$ to r_o in EF with respect to SF.

A possible way of detecting such a time delay is displayed in Fig. 3 where one considers a source of light at $r = r_s$ which emits both towards the observer placed at r_o (ray 1) and towards the black hole (ray 2). The latter ray then bounces back at r_b and reaches the observer with a delay with respect to ray 1 given by twice the time it takes to go from the source to r_b (this is a simple model for the so called *reverberation* in the accreting disk, see [21]). In EF this delay is given by (again assuming RND with $a = 1$)

$$\tau]_{EF} \sim 2\ \left[r_s - r_b + r_+\ \ln\left(\frac{r_s - r_+}{r_b - r_+}\right)\right]\ , \tag{4.12}$$

while in SF one has

$$\tau]_{SF} \sim \tau]_{EF} - 2\,r_-\ \ln\left(\frac{r_s - r_+}{r_b - r_+}\right)\ . \tag{4.13}$$

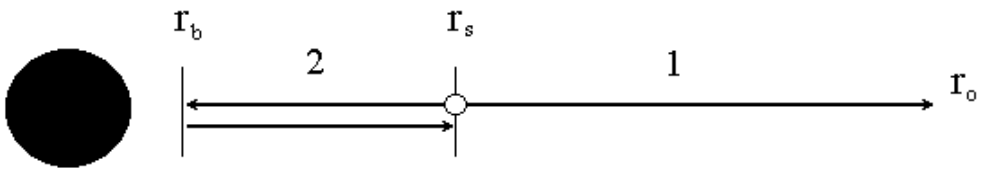

FIG. 3. Simple model of *reverberation*. Explanation is given in the text.

The difference between the metrics in the two frames also affects the red-shift z of waves emitted at r_s. For instance, in RND with $a = 1$ one has

$$z]_{EF} = -\frac{r_+}{r_s} = -\frac{2\,M}{r_s}$$
$$z]_{SF} = z]_{EF} - \frac{r_-}{r_s} + \frac{r_-\,r_+}{r_s^2} = z]_{EF} - \frac{Q^2}{r_s}\left(\frac{1}{M} - \frac{2}{r_s}\right) . \tag{4.14}$$

Finally, there are effects that cannot be accounted for without fully considering the wave nature of electromagnetic radiation. In fact, the electromagnetic waves couple directly to the dilaton (as can be seen in the action, see also Refs. [16,20] for explicit expressions in EF) and, although at leading orders the eikonal trajectories look the same in both frames, the intensity of the waves is different because of the different metric backgrounds. From (1.10) one can estimate the intensity of electromagnetic radiation produced in scattering events involving other fields in the model according to $I \sim |\vec{E}|^2 + |\vec{B}|^2 \sim F^2$. Therefore from (1.10) one gets

$$[I]_{SF} \sim e^{-2\,\phi}\ \left[F^2\right]_{EF} \sim e^{-2\,\phi}\,[I]_{EF} . \tag{4.15}$$

This implies $[I]_{SF} \sim [I]_{EF}\ \left(1 + \mathcal{O}(r^{-1})\right)$ and the difference is not appreciable when the scatterings occur far away from the hole.

V. CONCLUSION

The resolution of the issue of which frame is more suitable for the description of physical processes will have to wait at least until the next generation of astrophysical instruments. The measurement of the gyromagnetic ratio of a black hole will be a sensitive test of SF vs. EF, but precise measurements of the magnetic moment, the mass, the charge and the angular momentum of the black hole are necessary for the determination of this ratio. Existing telescopes are incapable of making such measurements, but future instruments will no doubt be able to 'see' close enough to the horizon of the black hole to determine these quantities with sufficient precision. Measurement of the intensity of electromagnetic radiation passing near the horizon of a black hole would also, in principle, be a very sensitive way of distinguishing between the two frames. This would require measuring the intensity of light from the stellar component of a black hole binary system for two different locations of the star – once when the star is occulting and once when it is eclipsing the black hole. The mass and charge of the black hole would also have to be known, as well as the angular momentum if frame dragging effects are to be taken into account. Although such binary systems are known to exist, precise measurement of the appropriate parameters is beyond present capabilities and will have to wait until the next generation of instruments become operative.

ACKNOWLEDGMENTS

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Conformal transformations in classical gravitational theories and in cosmology

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Abstract

In recent years, the use of conformal transformation techniques has become widespread in the literature on gravitational theories alternative to general relativity, on cosmology, and on nonminimally coupled scalar fields. Tipically, the transformation to the Einstein frame is generated by a fundamental scalar field already present in the theory. In this context, the problem of which conformal frame is the physical one has to be dealt with and, in the general case, it has been clarified only recently; the formulation of a theory in the "new" conformal frame leads to departures from canonical Einstein gravity. In this article, we review the literature on conformal transformations in classical gravitational theories and in cosmology, seen both as purely mathematical tools and as maps with physically relevant aspects. It appears particularly urgent to refer the analysis of experimental tests of Brans–Dicke and scalar–tensor theories of gravity, as well as the predictions of cosmological inflationary scenarios, to the physical conformal frame, in order to have a meaningful comparison with the observations.

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- 2. Conformal transformations as a mathematical tool
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- 6. Experimental consequences of the Einstein frame reformulation of gravitational theories
- 7. Nonminimal coupling of the scalar field
- 8. Conclusions
- Acknowledgments
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Notations and conventions

The notations and conventions used in this paper are as follows: the metric signature is – + + ... +. To facilitate the comparison with the literature on inflationary cosmology, we use units in which the speed of light and the reduced Planck constant assume the value unity. G is Newton's constant and the Planck mass is $m_{pl} = G^{-1/2}$ in these units. Greek indices assume the values 0, 1, 2, ..., $n-1$, where n is the dimension of spacetime. When $n = 4$, small Latin indices assume the values 1, 2, 3. While we allow for n spacetime dimensions (only one of which is timelike), in most of this paper the value $n = 4$ is assumed, except when discussing Kaluza–Klein and string theories prior to compactification.

A comma denotes ordinary differentiation, and ∇_μ is the covariant derivative operator. Round and square brackets around indices denote, respectively, symmetrization and antisymmetrization, which include division by the number of permutations of the indices: e.g. $A_{(\mu\nu)} = (A_{\mu\nu} + A_{\nu\mu})/2$. The Riemann and Ricci tensors are given in terms of the Christoffel symbols $\Gamma^\delta_{\alpha\beta}$ by

$$R_{\alpha\beta\gamma}{}^\delta = \Gamma^\delta_{\alpha\gamma,\beta} - \Gamma^\delta_{\beta\gamma,\alpha} + \Gamma^\sigma_{\alpha\gamma}\Gamma^\delta_{\sigma\beta} - \Gamma^\sigma_{\beta\gamma}\Gamma^\delta_{\sigma\alpha} \ ,$$

$$R_{\mu\rho} = \Gamma^\nu_{\mu\rho,\nu} - \Gamma^\nu_{\nu\rho,\mu} + \Gamma^\alpha_{\mu\rho}\Gamma^\nu_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\nu_{\alpha\mu} \ ,$$

and $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci curvature. $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is d'Alembert's operator. A tilde denotes quantities defined in the Einstein frame, while a caret denotes quantities defined in a higher–dimensional space prior to the compactification of the extra dimensions.

1 Introduction

If $(M, g_{\mu\nu})$ is a spacetime, the point–dependent rescaling of the metric tensor

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \ , \tag{1.1}$$

where $\Omega = \Omega(x)$ is a nonvanishing, regular function, is called a *Weyl* or *conformal transformation*. It affects the lengths of time [space]–like intervals and the norm of time [space]–like vectors, but it leaves the light cones unchanged: the spacetimes $(M, g_{\mu\nu})$ and $(M, \tilde{g}_{\mu\nu})$ have the same causal structure. The converse is also true (Wald 1984). If v^μ is a null, timelike, or spacelike vector with respect to the metric $g_{\mu\nu}$, it is also a null, timelike, or spacelike vector, respectively, in the rescaled metric $\tilde{g}_{\mu\nu}$.

Denoting by g the determinant $\det(g_{\mu\nu})$ one has, under the action of (1.1), $\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$ and $\tilde{g} \equiv \det(\tilde{g}_{\mu\nu}) = \Omega^{2n} g$. It will be useful to remember the transformation properties of the Christoffel symbols, Riemann and Ricci tensor, and of the Ricci curvature R under the rescaling (1.1) (Synge 1955; Birrell and Davies 1982; Wald 1984):

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \Omega^{-1} \left(\delta^{\alpha}_{\beta} \nabla_{\gamma} \Omega + \delta^{\alpha}_{\gamma} \nabla_{\beta} \Omega - g_{\beta\gamma} \nabla^{\alpha} \Omega \right) \, , \tag{1.2}$$

$$\widetilde{R_{\alpha\beta\gamma}}^{\delta} = R_{\alpha\beta\gamma}{}^{\delta} + 2\delta^{\delta}_{[\alpha} \nabla_{\beta]} \nabla_{\gamma} (\ln \Omega) - 2 g^{\delta\sigma} g_{\gamma[\alpha} \nabla_{\beta]} \nabla_{\sigma} (\ln \Omega) + 2 \nabla_{[\alpha} (\ln \Omega) \delta^{\delta}_{\beta]} \nabla_{\gamma} (\ln \Omega)$$
$$- 2 \nabla_{[\alpha} (\ln \Omega) g_{\beta]\gamma} g^{\delta\sigma} \nabla_{\sigma} (\ln \Omega) - 2 g_{\gamma[\alpha} \delta^{\delta}_{\beta]} g^{\sigma\rho} \nabla_{\sigma} (\ln \Omega) \nabla_{\rho} (\ln \Omega) \, , \tag{1.3}$$

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} - (n-2) \nabla_{\alpha} \nabla_{\beta} (\ln \Omega) - g_{\alpha\beta} g^{\rho\sigma} \nabla_{\rho} \nabla_{\sigma} (\ln \Omega) + (n-2) \nabla_{\alpha} (\ln \Omega) \nabla_{\beta} (\ln \Omega)$$
$$- (n-2) g_{\alpha\beta} \, g^{\rho\sigma} \nabla_{\rho} (\ln \Omega) \nabla_{\sigma} (\ln \Omega) \, , \tag{1.4}$$

$$\tilde{R} \equiv \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} = \Omega^{-2} \left[R - 2 (n-1) \, \Box \, (\ln \Omega) - (n-1) (n-2) \, \frac{g^{\alpha\beta} \nabla_{\alpha} \Omega \nabla_{\beta} \Omega}{\Omega^2} \right] \, , \tag{1.5}$$

where n ($n \geq 2$) is the dimension of the spacetime manifold M. In the case $n = 4$, the scalar curvature has the expressions

$$\tilde{R} = \Omega^{-2} \left[R - \frac{6 \Box \Omega}{\Omega} \right] = \Omega^{-2} \left[R - \frac{12 \Box (\sqrt{\Omega})}{\sqrt{\Omega}} - \frac{3 g^{\alpha\beta} \nabla_{\alpha} \Omega \nabla_{\beta} \Omega}{\Omega^2} \right] \, , \tag{1.6}$$

which are useful in many applications. The Weyl tensor $C_{\alpha\beta\gamma}{}^{\delta}$ (beware of the position of the indices !) is conformally invariant:

$$\widetilde{C_{\alpha\beta\gamma}}^{\delta} = C_{\alpha\beta\gamma}{}^{\delta} \, , \tag{1.7}$$

and the null geodesics are also conformally invariant (Lorentz 1937). The conservation equation $\nabla^{\nu} T_{\mu\nu} = 0$ for a symmetric stress–energy tensor $T_{\mu\nu}$ is not conformally invariant unless the trace $T \equiv T^{\mu}{}_{\mu}$ vanishes (Wald 1984). The Klein–Gordon equation $\Box \phi = 0$ for a scalar field ϕ is not conformally invariant, but its generalization

$$\Box \phi - \frac{n-2}{4(n-1)} \, R \, \phi = 0 \tag{1.8}$$

($n \geq 2$) is conformally invariant (note that the introduction of a nonzero cosmological constant in the Einstein action for gravity creates an effective mass, and a length scale,

in the Klein–Gordon equation, which spoils the conformal invariance (Madsen 1993)). Maxwell's equations in four dimensions are conformally invariant (Cunningham 1909; Bateman 1910), but the equations for the electromagnetic four–potential are not (it is to be noted that, at the quantum level, the conformal invariance of the Maxwell equations may be broken by quantum corrections like the generation of mass or the conformal anomaly). The conditions for conformal invariance of fields of arbitrary spin in any spacetime dimensions were discussed in (Iorio *et al.* 1997).

In this review paper, we will limit ourselves to consider special conformal transformations, in which the dependence of the conformal factor $\Omega(x)$ on the spacetime point x is obtained via a functional dependence (usually a power law) on a scalar field $\phi(x)$ present in the theory:

$$\Omega(x) = \Omega\left[\phi(x)\right] \ . \tag{1.9}$$

A redefinition of the scalar field ϕ accompanies the conformal transformation (1.1). Theories in which a fundamental scalar field appears and generates (1.1) include scalar–tensor and nonlinear theories of gravity (in which ϕ is a Brans–Dicke–like field) and Kaluza–Klein theories (in which ϕ is the determinant of the metric of the extra compact dimensions). Fundamental scalar fields in quantum theories include $SO(N)$ bosons in dual models, Nambu–Goldstone bosons, Higgs fields, and dilatons in superstring theories. In addition, almost all[1] scenarios of cosmological inflation (Linde 1990; Kolb and Turner 1990; Liddle and Lyth 1993; Liddle 1996) are based on scalar fields, either in the context of a classical or high energy theory, or in a phenomenological approach in which a scalar field is introduced as a source of gravitation in the field equations of the theory (usually the Einstein equations of general relativity). By means of a transformation of the form (1.1), many of these scenarios are recast in the form of Einstein gravity with the scalar field(s) as a source of gravity and a power–law inflationary potential. The investigation of this mathematical equivalence has far–reaching consequences, and in many cases the mathematical equivalence provides a means to go from a physically inconsistent theory to a viable one. Unfortunately, the use of conformal transformations in gravitational theories is haunted by confusion and ambiguities, particularly in relation to the problem of identifying the conformal frame which correctly describes the physics. Despite early work on the subject, confusion still persists in the literature and considerably detracts from papers that use conformal techniques incorrectly.

[1]The exception is R^2 inflation (Starobinsky 1980; Starobinsky 1986; Maeda, Stein–Schabes and Futamase 1989), in which the Lagrangian term R^2 itself drives inflation. However, a scalar field is sometimes added to this scenario to "help" inflation (Maeda 1989; Maeda, Stein–Schabes and Futamase 1989) and the scenario is often recast as power–law inflation by using a conformal transformation (Liddle and Lyth 1993).

It must be stressed that, in general, conformal transformations are not diffeomorphisms of the manifold M, and the rescaled metric $\tilde{g}_{\mu\nu}$ is not simply the metric $g_{\mu\nu}$ written in a different coordinate system: the metrics $\tilde{g}_{\mu\nu}$ given by Eq. (1.1) and the metric $g_{\mu\nu}$ describe different gravitational fields and different physics. Special conformal transformations originating from diffeomorphisms are called *conformal isometries* (Wald 1984). The reader should not be confused by the fact that some authors use the name "conformal transformation" for special coordinate transformations relating inertial and accelerated observers (e.g. Fulton, Rorlich and Witten 1962*a,b*; Wood, Papini and Cai 1989; Mashoon 1993). In this case the metric is left unchanged, although its coordinate representation varies. The possibility of different conformal rescalings for different metric components has also been considered (Mychelkin 1991), although it appears doubtful that this procedure can be given a covariant formulation and a physically sound motivation.

Historically, interest in conformal transformations arose after the formulation of Weyl's (1919) theory aimed at unifying gravitation and electromagnetism, expecially after its reformulation by Dirac (1973). Moreover, a conformally invariant version of special relativity was formulated (Page 1936*a,b*; Page and Adams 1936), but the conformal invariance in this case was recognized to be meaningless (Pauli 1958). Further developments of Weyl's theory are more appealing; for example, the self–consistent, scale–invariant theory of Canuto *et al.* (1977), so far, is not in contraddiction with the observations. It requires that the astronomical unit of length is related to the atomic unit by a scalar function which depends on the spacetime point. The theory contains a time–dependent cosmological "constant" $\Lambda(t) = \Lambda_0(t_0/t)^2$, which is sought after by many authors in modern cosmology and astroparticle physics.

2 Conformal transformations as a mathematical tool

Conformal rescalings and conformal techniques have been widely used in general relativity for a long time, expecially in the theory of asymptotic flatness and in the initial value formulation (Wald 1984 and references therein), and also in studies of the propagation of massless fields, including Fermat's principle (Perlick 1990; Schneider, Ehlers and Falco 1992), gravitational lensing in the (conformally flat) Friedmann–Lemaitre–Robertson–Walker universe (Perlick 1990; Schneider, Ehlers and Falco 1992), wave equations (Sonego and Faraoni 1992; Noonan 1995), studies of the optical geometry near black hole horizons (Abramowicz, Carter and Lasota 1988; Sonego and Massar 1996; Abramowicz *et al.* 1997*a,b*), exact solutions (Van den Bergh 1986*a,b,c,d,e*, 1988)

and in other contexts. Conformal techniques and conformal invariance are important also for quantum field theory in curved spaces (Birrell and Davies 1982), for statistical mechanics and for string theories (e.g. Dita and Georgescu 1989). A conformal transformation is often used as a mathematical tool to map the equations of motion of physical systems into mathematically equivalent sets of equations that are more easily solved and computationally more convenient to study. This situation arises mainly in three different areas of gravitational physics: alternative (including nonlinear) theories of gravity, unified theories in multidimensional spaces, and studies of scalar fields nonminimally coupled to gravity.

Brans–Dicke theory: The conformal rescaling to the minimally coupled case for the Brans–Dicke field in Brans–Dicke theory was found by Dicke (1962). One starts with the Brans–Dicke action in the so–called "Jordan frame"

$$S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\phi\, R - \frac{\omega}{\phi}\, \nabla^\mu \phi \nabla_\mu \phi \right] + S_{matter} \ , \tag{2.1}$$

which corresponds to the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right) + \frac{1}{\phi} \left(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi \right) \ , \tag{2.2}$$

$$\Box \Phi = \frac{8\pi T}{3 + 2\omega} \ . \tag{2.3}$$

The conformal transformation (1.1) with

$$\Omega = \sqrt{G\phi} \tag{2.4}$$

and the redefinition of the scalar field given in differential form by

$$d\tilde{\phi} = \sqrt{\frac{2\omega + 3}{16\pi G}}\ \frac{d\phi}{\phi} \tag{2.5}$$

($\omega > -3/2$) transform the action (2.1) into the "Einstein frame" action

$$S = \int d^4x \left\{ \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}_\mu \tilde{\phi} \right] + \exp\left(-8 \sqrt{\frac{\pi G}{2\omega + 3}} \tilde{\phi} \right) \mathcal{L}_{matter}(\tilde{g}) \right\} \ , \tag{2.6}$$

where $\tilde{\nabla}_\mu$ is the covariant derivative operator of the rescaled metric $\tilde{g}_{\mu\nu}$. The gravitational part of the action now contains only Einstein gravity, but a free scalar field acting

as a source of gravitation *always* appears. It permeates spacetime in a way that cannot be eliminated, i.e. one cannot contemplate solutions of the vacuum Einstein equations $R_{\mu\nu} = 0$ in the Einstein frame. In the Jordan frame, the gravitational field is described by the metric tensor $g_{\mu\nu}$ *and* by the Brans–Dicke field ϕ. In the Einstein frame, the gravitational field is described only by the metric tensor $\tilde{g}_{\mu\nu}$, but the scalar field $\tilde{\phi}$, which is now a form of matter, is always present, a reminiscence of its fundamental role in the "old" frame. In addition, the rest of the matter part of the Lagrangian is multiplied by an exponential factor, thus displaying an anomalous coupling to the scalar $\tilde{\phi}$. This anomalous coupling will be discussed in Sec. 6.

Nonminimally coupled scalar field: By means of a conformal rescaling, the study of a nonminimally coupled scalar field can also be reduced to that of a minimally coupled scalar. The transformation relating a massless conformally coupled and a minimally coupled scalar field was found by Bekenstein (1974) and later rediscovered and generalized to massive fields and arbitrary values of the coupling constant (Deser 1984; Schmidt 1988; Maeda 1989; Futamase and Maeda 1989; Xanthopoulos and Dialynas 1992; Klimcik 1993; Accioly *et al.* 1993). In this case, the starting point is the action for canonical gravity plus a scalar field in the Jordan frame:

$$S = \int d^4x\sqrt{-g}\left[\left(\frac{1}{16\pi G} - \frac{\xi\phi^2}{2}\right)R - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right] , \tag{2.7}$$

where $V(\phi)$ is the scalar field potential (possibly including a mass term and the cosmological constant) and ξ is a dimensionless coupling constant. Note that the dimensions of the scalar field are $[\phi] = \left[G^{-1/2}\right] = [m_{pl}]$. The equation satisfied by the scalar ϕ is

$$\Box\phi - \xi R\phi - \frac{dV}{d\phi} = 0 . \tag{2.8}$$

Two cases occur most frequently in the literature: "minimal coupling" ($\xi = 0$) and "conformal coupling" ($\xi = 1/6$); the latter makes the wave equation (2.8) conformally invariant in four dimensions if $V = 0$ or $V = \lambda\phi^4$ (the latter potential being used in the chaotic inflationary scenario). The conformal transformation (1.1) with

$$\Omega^2 = 1 - 8\pi G\xi\phi^2 \tag{2.9}$$

and the redefinition of the scalar field, given in differential form by

$$d\tilde{\phi} = \frac{\left[1 - 8\pi G\xi\left(1 - 6\xi\right)\phi^2\right]^{1/2}}{1 - 8\pi G\xi\phi^2}\, d\phi , \tag{2.10}$$

reduce (2.7) to the Einstein frame action

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{\nabla}^\mu\tilde{\phi}\tilde{\nabla}_\mu\tilde{\phi} - \tilde{V}(\tilde{\phi})\right] , \tag{2.11}$$

where the scalar field $\tilde{\phi}$ is now minimally coupled and satisfies the equation

$$\tilde{g}^{\mu\nu}\nabla_\mu\nabla_\nu\tilde{\phi} - \frac{d\tilde{V}}{d\tilde{\phi}} = 0 . \tag{2.12}$$

The new scalar field potential is given by

$$\tilde{V}(\tilde{\phi}) = \frac{V(\phi)}{\left(1 - 8\pi G\xi\phi^2\right)^2} , \tag{2.13}$$

where $\phi = \phi\left(\tilde{\phi}\right)$ is obtained by integrating and inverting Eq. (2.10). The field equations of a gravitational theory in the case of a minimally coupled scalar field as a source of gravity are computationally much easier to solve than the corresponding equations for nonminimal coupling, and the transformation (1.1), (2.9), (2.10) is widely used for this purpose. The stress–energy tensor of a scalar field can be put in the form corresponding to a fluid, but the $T_{\mu\nu}$ for a nonminimally coupled field is considerably more complicated than the minimal coupling case, for which the form of the $T_{\mu\nu}$ reduces to that of a perfect fluid (Madsen 1988). It is generally assumed that the scalar field ϕ assumes values in a range that makes the right hand side of Eq. (2.9) positive. For $\xi > 0$, this range is limited by the critical values $\phi_{1,2} = \pm\left(8\pi G\xi\right)^{-1/2}$.

Nonminimal couplings of the electromagnetic field to gravity have also been considered (Novello and Salim 1979; Novello and Heintzmann 1984; Turner and Widrow 1988; Novello and Elbaz 1994; Novello, Pereira and Pinto–Neto 1995; Lafrance and Myers 1995), but conformal techniques analogous to those developed for scalar fields are presently unknown. A formal method alternative to conformal transformations is sometimes useful for nonminimally coupled scalar fields, which are equivalent to an effective flat space field theory with a scalar mass that is ξ–dependent (Hochberg and Kephart 1995).

Nonlinear theories of gravity: The mathematical equivalence between a theory described by the gravitational Lagrangian density $\mathcal{L}_g = \sqrt{-g}f(R)$ ("higher order theory") and Einstein gravity was found in (Teyssandier and Tourrenc 1983; Schmidt 1987;

Starobinsky 1987; Barrow and Cotsakis 1988; Maeda 1989; Gott, Schmidt and Starobinsky 1990; Schmidt 1990; Cotsakis and Saich 1994; Wands 1994). The field equations for this theory are of fourth order:

$$\left(\frac{df}{dR}\right) R_{\mu\nu} - \frac{1}{2}\, f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu \left(\frac{df}{dR}\right) + g_{\mu\nu} \Box \left(\frac{df}{dR}\right) = 0 \ , \tag{2.14}$$

and are reduced to the Einstein equations by the conformal transformation.

Quadratic Lagrangian densities with R^2 terms arising from quantum corrections are the most frequently studied cases of nonlinear gravitational theories; they can be reduced to the Einstein Lagrangian density (Higgs 1959; Teyssandier and Tourrenc 1983; Whitt 1984; Ferraris 1986; Berkin and Maeda 1991). These results were generalized to supergravity, Lagrangians with terms $\Box^k R$ ($k \geq 1$) and polynomial Lagrangians in R (Cecotti 1987); the two–dimensional case was studied in (Mignemi and Schmidt 1995). This class of theories includes Weyl's theory (Weyl 1919; Dirac 1973) described by the Lagrangian density $\mathcal{L} = \sqrt{-g}(R^2 + \beta F_{\mu\nu} F^{\mu\nu})$, and theories of the form $\mathcal{L} = R^k$ ($k \geq 1$). For nonlinear theories of gravity, the conformal transformation that maps the theory into Einstein gravity becomes a Legendre transformation (Ferraris, Francaviglia and Magnano 1988; Jakubiec and Kijowski 1988; Ferraris, Francaviglia and Magnano 1990; Magnano, Ferraris and Francaviglia 1990; Magnano and Sokolowski 1994).

There are obvious advantages in performing this transformation because the higher order field equations of the nonlinear theory are reduced to the second order Einstein equations with matter. One starts with a purely gravitational nonlinear theory described by the action

$$S = \int d^m x \sqrt{-g} \left[F(\phi, R) - \frac{\epsilon}{2} \nabla^\mu \phi \nabla_\mu \phi \right] \ , \tag{2.15}$$

in m spacetime dimensions, where $F(\phi, R)$ is an arbitrary (but sufficiently regular) function of ϕ and R, and ϵ is a free parameter (normally 0 or 1).

The corresponding field equations (Maeda 1989) are

$$\begin{aligned} \left(\frac{\partial F}{\partial R}\right) \left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\right) \ &= \ \frac{\epsilon}{2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right) + \frac{1}{2} g_{\mu\nu} \left(F - \frac{\partial F}{\partial R} R \right) \\ &\quad + \nabla_\mu \nabla_\nu \left(\frac{\partial F}{\partial R}\right) - g_{\mu\nu} \Box \left(\frac{\partial F}{\partial R}\right) \ , \end{aligned} \tag{2.16}$$

$$\epsilon \Box \phi = - \frac{\partial F}{\partial \phi} \ . \tag{2.17}$$

The conformal rescaling (1.1), where

$$\Omega^2 = \left[16\pi G \left|\frac{\partial F}{\partial R}\right| + \text{constant}\right]^{2/(m-2)} , \tag{2.18}$$

and the redefinition of the scalar field

$$\tilde{\phi} = \frac{1}{\sqrt{8\pi G}} \sqrt{\frac{m-1}{m-2}} \ln\left[\sqrt{32\pi} G \left|\frac{\partial F}{\partial R}\right|\right] \tag{2.19}$$

(Maeda 1989) reduce the action (2.15) to

$$S = \alpha \int d^m x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}_\mu \tilde{\phi} - \frac{\epsilon\alpha}{2} \exp\left[-\sqrt{8\pi G \, \frac{m-2}{m-1}} \, \tilde{\phi}\right] - U(\phi, \tilde{\phi})\right] \tag{2.20}$$

where the two scalar fields ϕ and $\tilde{\phi}$ appear and

$$\alpha = \frac{\partial F/\partial R}{|\partial F/\partial R|} , \tag{2.21}$$

$$U(\phi, \tilde{\phi}) = \alpha \exp\left(-\frac{m\sqrt{8\pi G}\tilde{\phi}}{\sqrt{(m-1)(m-2)}}\right)\left[\frac{\alpha}{16\pi G} R(\phi, \tilde{\phi}) \exp\left(\sqrt{\frac{m-2}{m-1} 8\pi G} \, \tilde{\phi}\right) - F(\phi, \tilde{\phi})\right] , \tag{2.22}$$

and $F(\phi, \tilde{\phi}) = F(\phi, R(\phi, \tilde{\phi}))$. The resulting system is of nonlinear σ–model type, canonical gravity with two scalar fields ϕ, $\tilde{\phi}$.

In the particular case in which $F(\phi, R)$ is a linear function of the Ricci curvature,

$$F(\phi, R) = f(\phi)R - V(\phi) , \tag{2.23}$$

the redefinition of the scalar field

$$\tilde{\phi} = \frac{1}{\sqrt{8\pi G}} \int d\phi \left\{\frac{\epsilon(m-2)f(\phi) + 2(m-1)\left[df(\phi)/d\phi\right]^2}{2(m-2)f^2(\phi)}\right\}^{1/2} \tag{2.24}$$

(where the argument of the square root is assumed to be positive) leads to the Einstein action with a single scalar field ϕ:

$$S = \frac{|f|}{f} \int d^m x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}_\mu \tilde{\phi} - U(\tilde{\phi})\right] . \tag{2.25}$$

This action is equivalent to the Einstein equations

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = 8\pi G\, \tilde{T}_{\mu\nu}\left[\tilde{\phi}\right] \ , \tag{2.26}$$

$$\tilde{T}_{\mu\nu}\left[\tilde{\phi}\right] = \nabla_\mu\tilde{\phi}\nabla_\nu\tilde{\phi} - \frac{1}{2}\,\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\nabla_\alpha\tilde{\phi}\nabla_\beta\tilde{\phi} + U\,\tilde{g}_{\mu\nu} \ , \tag{2.27}$$

where

$$U(\tilde{\phi}) = \frac{|f|}{f}\left[16\pi G\,|f(\phi)|\right]^{\frac{-m}{m-2}}\ V(\phi) \tag{2.28}$$

and $\phi = \phi\left(\tilde{\phi}\right)$. The transformations (2.4), (2.9) and (2.10) are recovered as particular cases of (2.24), (2.18). In addition, all the theories described by a four–dimensional action of the form

$$S = \int d^4x\sqrt{-g}\,[f(\phi)R + A(\phi)\nabla^\mu\phi\nabla_\mu\phi + V(\phi)] \tag{2.29}$$

and satisfying the relation

$$2Af - 3\left(\frac{df}{d\phi}\right)^2 = 0 \ , \qquad V(\phi) = \lambda f^2(\phi) \tag{2.30}$$

(λ =constant) are conformally related (Shapiro and Takata 1995); particular cases include general relativity and the case of a conformally coupled scalar field.

The conformal transformation establishes a *mathematical* equivalence between the theories formulated in the two conformal frames; the space of solutions of the theory in one frame is isomorphic to the space of solutions in the conformally related frame (which is mathematically more convenient to study). The conformal transformation can also be used as a solution–generating technique, if solutions are known in one conformal frame but not in another (Harrison 1972; Belinskii and Kalatnikov 1973; Bekenstein 1974; Van den Bergh 1980, 1982, 1983*a,b,c,d*; Froyland 1982; Accioly, Vaidya and Som 1983; Lorentz–Petzold 1984; Barrow and Maeda 1990; Klimcik and Kolnik 1993; Abreu, Crawford and Mimoso 1994). It is to be stressed that the mathematical equivalence between the two systems *a priori* implies nothing about their physical equivalence (Brans 1988; Cotsakis 1993; Magnano and Sokolowski 1994). Moreover, only the gravitational (vacuum) part of the action is conformally equivalent to Einstein gravity: if ordinary matter (i.e. matter different from the scalar field used in the conformal transformation) is added to the theory, the coupling of this matter to gravity and the conservation equations that it satisfies are different in the two conformally related frames. The advantage of the

conformal transformation in a non–purely vacuum theory is questionable: it has been argued that, because the Einstein frame scalar field is coupled to matter, a simplification of the equations of motion in this case does not occur (Barrow and Maeda 1990).

Not only it is possible to map the classes of theories considered above into canonical Einstein gravity, but it is also possible to find conformal transformations between each two of these theories (see Magnano and Sokolowski 1994 for a table of possible transformations). Indeed, one expects to be able to do that by taking appropriate compositions of different maps from gravitational theories to general relativity, and their inverse maps.

We conclude this section with a remark on the terminology: it has become common to use the word "frame" to denote a set of dynamical variables of the theory considered; the term "gauge" instead of "frame" has been (rather improperly) used (Gibbons and Maeda 1988; Brans 1988). In some papers (Cho 1987, 1990, 1992, 1994, 1997; Cho and Yoon 1993) the metric $\tilde{g}_{\mu\nu}$ in the Einstein frame is called "Pauli metric", as opposed to the "Jordan" or "atomic unit" metric $g_{\mu\nu}$ of the Jordan frame.

3 Is the Einstein frame physical ?

Many high energy theories and many classical gravity theories are formulated by using a conformal transformation mapping the Jordan frame to the Einstein frame. Typically, the conformal factor of the transformation is a function of a dilaton, or Brans–Dicke–like field already present in the theory. The classical theories of gravity for which a conformal transformation maps the system into a new conformal frame, in which the gravitational sector of the theory reduces to the canonical Einstein form, include Brans–Dicke theory and its scalar–tensor generalizations, non–linear gravity theories, classical Kaluza–Klein theories and in general, all theories which have an extended gravitational sector or which involve a dimensional reduction and compactification of extra spacetime dimensions. Quantum theories incorporating the conformal transformation include superstring and supergravity theories and σ–models. The transformation to the Einstein frame seems to be universally accepted for supergravity and superstring theories (although field redefinitions may be an issue for debate (Tseytlin 1993)). It is unknown whether physics is conformally invariant at a sufficiently high energy scale, but there are indications in this sense from string theories (Green, Schwarz and Witten 1987) and from $SU(N)$ induced gravity models in which, in the high energy limit, the scalar fields of the theory approach conformal coupling (Buchbinder, Odintsov and Shapiro 1992; Geyer and Odintsov 1996). We have no experiments capable of probing the energy scale of string theories, and conformal invariance at this energy scale cannot be directly

tested. While the low–energy Einstein gravity contains a dynamical degree of freedom connected with the "length" of the metric tensor (the determinant g), this is absent in conformally invariant gravity (e.g. induced gravity described by the action (4.17)). The conformal invariance of a theory implies that the latter contains no intrinsic mass; a nonzero mass would introduce a preferred length scale in the theory, thus breaking the scale–invariance. The physical inequivalence of conformal frames at low energies reflects the fact that the non–negligible masses of the fields present in the theory break the conformal symmetry which is present at higher energies. In classical gravity theories, there is disagreement and confusion on the long–standing (Pauli 1955; Fierz 1956) problem of which conformal frame is the physical one. Is the Jordan frame physical and the Einstein frame unphysical ? Is the conformal transformation necessary, and the Einstein frame physical ? Does any other choice of the conformal factor in Eq. (1.1) map the theory into a physically significant frame, and how many of these theories are possible ? Here the term "physical" theory denotes one that is theoretically consistent and predicts the values of some observables that can, at least in principle, be measured in experiments performed in four macroscopic spacetime dimensions (definitions that differ from ours are sometimes adopted in the literature, see e.g. (Garay and Garcia–Bellido 1993; Overduin and Wesson 1997)). The ambiguity in the choice of the physical conformal frame raises also problems of an almost philosophical character (Weinstein 1996).

Before attempting to answer any of these questions, it is important to recognize that, in general, the reformulation of the theory in a new conformal frame leads to a different, physically inequivalent theory. If one restricts oneself to consider the metric tensor and physics that does not involve only conformally invariant fields (e.g. a stress–energy tensor $T_{\mu\nu}$ with nonvanishing trace), or experiments involving massive particles and timelike observers, it is obvious that metrics conformally related by a nontrivial transformation of the kind (1.1) on a manifold describe different gravitational fields and different physical situations. For example, one could consider a Friedmann–Lemaitre-Robertson–Walker metric with flat spatial sections, given by the line element

$$ds^2 = a^2(\eta)\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right) \ , \tag{3.1}$$

where η is the conformal time and (x, y, z) are spatial comoving coordinates. The metric (3.1) is conformally flat, but certainly it is not physically equivalent to the Minkowski metric $\eta_{\mu\nu}$, since it exhibits a nontrivial dynamics and significant (observed) cosmological effects.

The authors working in classical gravitational physics can be grouped into five categories according to their attitude towards the issue of the conformal frame (we partially follow a previous classification by Magnano and Sokolowski (1994)):

- authors that neglect the issue (Deruelle and Spindel 1990; Garcia–Bellido and Quirós 1990; Hwang 1990; Gottlöber, Müller and Starobinsky 1991; Suzuki and Yoshimura 1991; Rothman and Anninos 1991; Guendelman 1992; Guth and Jain 1992; Liddle and Wands 1992; Capozziello, Occhionero and Amendola 1993; Capozziello, de Ritis and Rubano 1993; McDonald 1993*a,b*; Barrow, Mimoso and de Garcia Maia 1993; Garcia–Bellido and Wands 1995; Laycock and Liddle 1994; Alvarez and Belén Gavela 1983; Sadhev 1984; Deruelle and Madore 1987; Van den Bergh and Tavakol 1993; Fabris and Sakellariadou 1997; Kubyshin and Martin 1995; Fabris and Martin 1993; Chatterjee and Banerjee 1993; Biesiada 1994; Liddle and Lyth 1993; Hwang 1996);

- authors that explicitely support the view that a theory formulated in one conformal frame is physically equivalent to the reformulation of the same theory in a different conformal frame (Buchmüller and Dragon 1989; Holman, Kolb and Wang 1990; Campbell, Linde and Olive 1991; Casas, Garcia–Bellido and Quirós 1991; Garay and Garcia–Bellido 1993; Levin 1995*a,b*; Shapiro and Takata 1995; Kaloper and Olive 1998);

- authors that are aware of the physical non–equivalence of conformally related frames but do not present conclusive arguments in favour of one or the other of the two versions of the theory (and/or perform computations both in the Jordan and the Einstein frame) (Brans 1988; Jakubiec and Kijowski 1988; Kasper and Schmidt 1989; Deruelle and Spindel 1990; Hwang 1990; Kolb, Salopek and Turner 1990; Gottlöber, Müller and Starobinsky 1991; Suzuki and Yoshimura 1991; Rothman and Anninos 1991; Guendelman 1992; Guth and Jain 1992; Liddle and Wands 1992; Piccinelli, Lucchin and Matarrese 1992; Damour and Nordvedt 1993*a,b*; Cotsakis and Saich 1994; Hu, Turner and Weinberg 1994; Turner 1993; Mimoso and Wands 1995*b*; Faraoni 1996*a*; Weinstein 1996; Turner and Weinberg 1997; Majumdar 1997; Capozziello, de Ritis and Marino 1997; Dick 1998);

- authors that identify the Jordan frame as physical (possibly allowing the use of the conformal transformation as a purely mathematical tool) (Gross and Perry 1983; Barrow and Maeda 1992; Berkin, Maeda and Yokoyama 1990; Damour, Gibbons and Gundlach 1990; Kalara, Kaloper and Olive 1990; Berkin and Maeda 1991; Damour and Gundlach 1991; Holman, Kolb and Wang 1991; Mollerach and Matarrese 1992; Tao and Xue 1992; Wu 1992; del Campo 1992; Tkacev 1992; Mignemi and Whiltshire 1992; Barrow 1993; Bruckman and Velazquez 1993; Cotsakis and Flessas 1993; Will and Steinhardt 1995; Scheel, Shapiro and Teukolsky

1995; Barros and Romero 1998);

- authors that identify the Einstein frame as the physical one (Van den Bergh 1981, 1983*e*; Kunstatter, Lee and Leivo 1986; Gibbons and Maeda 1988; Sokolowski 1989*a,b*; Pimentel and Stein–Schabes 1989; Kubyshin, Rubakov and Tkachev 1989; Salopek, Bond and Bardeen 1989; Kolb, Salopek and Turner 1990; Cho 1990; Deruelle, Garriga and Verdaguer 1991; Cho 1992; Amendola *et al.* 1992; Amendola, Bellisai and Occhionero 1993; Cotsakis 1993; Cho and Yoon 1993; Alonso *et al.* 1994; Magnano and Sokolowski 1994; Cho 1994; Occhionero and Amendola 1994; Lu and Cheng 1996; Fujii 1998; Cho 1997; Cho and Keum 1998).

Sometimes, works by the same author(s) belong to two different groups; this illustrates the confusion on the issue that is present in the literature.

The two conformal frames, however, are substantially different. Furthermore, if a preferred conformal frame does not exist, it is possible to generate an infinite number of alternative theories and of cosmological inflationary scenarios by arbitrarily choosing the conformal factor (1.9) of the transformation (1.1). Only when a physical frame is uniquely determined the theory and its observable predictions are meaningful.

Earlier attempts to solve the problem in Brans–Dicke theory advocated the equivalence principle: to this end it is essential to consider not only the gravitational, but also the matter part of the Lagrangian. The use of the equivalence principle requires a careful analysis (Brans 1988; Magnano and Sokolowski 1994); by including the Lagrangian for ordinary matter in the Jordan frame action, one finds that, after the conformal transformation has been performed, the scalar field in the Einstein frame couples minimally to gravity, but nonminimally to matter ("non–universal coupling"). Historically, the Jordan frame was selected as physical because the dilaton couples minimally to ordinary matter in this frame (Brans and Dicke 1961). Attempts were also made to derive conclusive results from the conservation laws for the stress–energy tensor of matter, favouring the Jordan frame (Brans 1988) or the Einstein frame (Cotsakis 1993; Cotsakis 1995 – see (Teyssandier 1995; Schmidt 1995; Magnano and Sokolowski 1994) for the correction of a flaw in the proof of (Cotsakis 1993; Cotsakis 1995)). Indeed, the conservation laws do not allow one to draw definite conclusions (Magnano and Sokolowski 1994).

However, the point of view that selects the Jordan frame as physical is untenable because it leads to a negative definite, or indefinite kinetic energy for the scalar field; on the contrary, the energy density is positive definite in the Einstein frame. This result was initially proved for Brans–Dicke and for Kaluza–Klein theories, and later generalized to gravitational theories with Lagrangian density $\mathcal{L} = f(R)\sqrt{-g} + \mathcal{L}_{matter}$ (Magnano and Sokolowski 1994). This implies that the theory does not have a stable ground state, and

that the system decays into a lower energy state *ad infinitum* (Gross and Perry 1983; Appelquist and Chodos 1983; Maeda 1986*b*; Maeda 1987). While a stable ground state may not be required for certain particular solutions of the theory (e.g. cosmological solutions (Padmanabhan 1988)), or for Liouville's theory (D'Hoker and Jackiw 1982), it is certainly necessary for a viable theory of classical gravity. The ground state of the system must be stable against small fluctuations and not fine–tuned, i.e. nearby solutions of the theory must have similar properties (Strater and Wightman 1964; Epstein, Glaser and Jaffe 1965; Abbott and Deser 1982). The fact that the energy is not positive definite is usually associated with the formulation of the theory in unphysical variables. On the contrary, the energy conditions (Wald 1984) are believed to be satisfied by all classical matter and fields (not so in quantum theories – see Sec. 8). This decisive argument was first used to select the Einstein frame in Kaluza–Klein and Brans–Dicke theories (Bombelli *et al.* 1987; Sokolowski and Carr 1986; Sokolowski 1989*a,b*; Sokolowski and Golda 1987; Cho 1990; Cho 1994), and later generalized to scalar–tensor theories (Cho 1997) and nonlinear gravity theories (Magnano and Sokolowski 1994). Also, the uniqueness of a physical conformal frame was proved (Sokolowski 1989*a,b*; Magnano and Sokolowski 1994).

For completeness, we mention other arguments supporting the Einstein frame as physical that have appeared in the literature: however, they are either highly questionable (sometimes to the point of not being valid), or not as compelling as the one based on the positivity of energy. The Hilbert and the Palatini actions for scalar–tensor theories are equivalent in the Einstein but not in the Jordan frame (Van den Bergh 1981, 1983*e*). Some authors choose the Einstein frame on the basis of the resemblance of its action with that of general relativity (Gibbons and Maeda 1988; Pimentel and Stein–Schabes 1989; Alonso *et al.* 1994; Amendola, Bellisai and Occhionero 1993); others (Salopek, Bond and Bardeen 1989; Kolb, Salopek and Turner 1990) find difficulties in quantizing the scalar field fluctuations in the linear approximation in the Jordan frame, but not in the Einstein frame; quantization and the conformal transformation do not commute (Fujii and Nishioka 1990; Nishioka and Fujii 1992; Fakir and Habib 1993). Other authors claim that the Einstein frame is forced upon us by the compactification of the extra dimensions in higher dimensional theories (Kubyshin, Rubakov and Tkachev 1989; Deruelle, Garriga and Verdaguer 1991).

A possible alternative to the Einstein frame formulation of the complete theory (gravity plus matter) has been supported (Magnano and Sokolowski 1994), and consists in starting with the introduction of matter non–minimally coupled to the Brans–Dicke scalar in the Jordan frame, with the coupling tuned in such a way that the Einstein frame action exhibits matter minimally coupled to the Einstein frame scalar field, after

the conformal transformation has been performed. This procedure arises from the observation (Magnano and Sokolowski 1994) that the traditional way of prescribing matter minimally coupled in the Jordan frame relies on the implicit assumptions that
i) the equivalence principle holds;
ii) the Jordan frame is the physical one.
While these assumptions are not justified *a priori*, as noted by Magnano and Sokolowski (1994), the possibility of adding matter in the Jordan frame with a coupling that exactly balances the exponential factor appearing in the Einstein frame appears to be completely *ad hoc* and is not physically motivated; by proceeeding along these lines, one could arbitrarily change the theory without theoretical justification.

As a summary, the Einstein frame is the physical one (and the Jordan frame and all the other conformal frames are unphysical) for the following classes of theories:

- scalar–tensor theories of gravity described by the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[f(\phi)R - \frac{\omega(\phi)}{\phi}\nabla^{\mu}\phi\nabla_{\mu}\phi + \Lambda(\phi)\right] + \mathcal{L}_{matter}\ , \tag{3.2}$$

  which includes Brans–Dicke theory as a special case (see Sec. 4 for the corresponding field equations);

- classical Kaluza–Klein theories;

- nonlinear theories of gravity whose gravitational part is described by the Lagrangian density $\mathcal{L} = \sqrt{-g}f(R)$ (see Sec. 4 for the corresponding field equations).

Since the Jordan frame formulation of alternative theories of gravity is unphysical, one reaches the conclusion that the Einstein frame formulation is the only possible one for a classical theory. In other words, this amounts to say that Einstein gravity is essentially the only viable classical theory of gravity (Bicknell 1974; Magnano and Sokolowski 1994; Magnano 1995; Sokolowski 1997). We remark that this statement is strictly correct only if the purely gravitational part of the action (without matter) is considered: in fact, when matter is included into the action, in general it exhibits an anomalous coupling to the scalar field which does not occur in general relativity.

Finally, we comment on the case of a nonminimally coupled scalar field described by the action (2.7). From the above discussion, one may be induced to believe that the Einstein frame description is necessary also in this case: this conclusion would be incorrect because the kinetic term in the action (2.7) is canonical and positive definite, and the problem discussed above for other theories of gravity of the form $\mathcal{L} = \sqrt{-g}f(R)$

does not arise. It is, however, still true that the Einstein and the Jordan frame are physically inequivalent: the conformal transformation (1.1), (2.9), (2.10) implies only a mathematical, not a physical equivalence, despite strong statements on this regard that point to the contrary (Accioly *et al.* 1993).

4 Conformal transformations in gravitational theories

In this section, we review in greater detail the arguments that led to the conclusions of the previous section, devoting more attention to specific classical theories of gravity.

Brans–Dicke theory: The Jordan–Fierz–Brans–Dicke theory (Jordan 1949; Jordan 1955; Fierz 1956; Brans and Dicke 1961) described by the action (2.1) (where ϕ has the dimensions of the inverse gravitational constant, $[\phi] = [G^{-1}]$) has been the subject of renewed interest, expecially in cosmology in the extended inflationary scenario (La and Steinhardt 1989; Kolb, Salopek and Turner 1990; Laycock and Liddle 1994). The recent surge of interest appears to be motivated by a restricted conformal invariance of the gravitational part of the Lagrangian that mimics the conformal invariance of string theories before conformal symmetry is broken (Cho 1992; Cho 1994; Turner 1993; Kolitch and Eardley 1995; Brans 1997; Cho and Keum 1998). The conformal transformation (1.1) with $\Omega = (G\phi)^{\alpha}$, together with the redefinition of the scalar field $\tilde{\phi} = G^{-2\alpha}\phi^{1-2\alpha}$ ($\alpha \neq 1/2$) maps the Brans–Dicke action (2.1) into an action of the same form, but with parameter

$$\tilde{\omega} = \frac{\omega - 6\alpha\,(\alpha - 1)}{(1 - 2\alpha)^2}\ . \tag{4.1}$$

If $\omega = -3/2$, the action (2.1) is invariant under the conformal transformation; this case corresponds to the singularity $\alpha \to -1/2$ in the expression (4.1), but the field equations (2.2), (2.3) are not defined in this case. This conformal invariance is broken when a term describing matter with $T \equiv T^{\mu}{}_{\mu} \neq 0$ is added to the purely gravitational part of the Brans–Dicke Lagrangian. This property of conformal invariance of the gravitational sector of the theory is enjoyed also by a subclass of more general tensor–multiscalar theories of gravity (Damour and Esposito–Farèse 1992) and has not yet been investigated in depth in the general case. The study of the conformal invariance property of Brans–Dicke theory helps to solve the problems arising in the $\omega \to \infty$ limit of Brans–Dicke theory (Faraoni 1998). This limit is supposed to give back general relativity, but it

fails to do so when $T = 0$. The differences between the Jordan and the Einstein frame formulations of Brans–Dicke theory have been pointed out clearly in (Guth and Jain 1992). It has been noted in studies of gravitational collapse to a black hole in Brans–Dicke theory that the noncanonical form of the Brans–Dicke scalar energy–momentum tensor in the Jordan frame violates the null energy condition ($R_{\alpha\beta}l^\alpha l^\beta \geq 0$ for all null vectors l^α). This fact is responsible for a *decrease* in time of the horizon area during the dynamical phase of the collapse, contrarily to the case of general relativity (Scheel, Shapiro and Teukolsky 1995). The violation of the weak energy condition in the Jordan frame has also been pointed out (Weinstein 1996; Faraoni and Gunzig 1998*a*).

Brans–Dicke theory must necessarily be reformulated in the Einstein frame; the strongest argument supporting this conclusion is obtained by observing that the kinetic energy term for the Brans–Dicke field in the Jordan frame Brans–Dicke Lagrangian does not have the canonical form for a scalar field, and it is negative definite (Gross and Perry 1983; Appelquist and Chodos 1983; Maeda 1986*b*; Maeda 1987; Sokolowski and Carr 1986; Maeda and Pang 1986; Sokolowski 1989*a,b*; Cho 1992, 1993; Magnano and Sokolowski 1994). The fact that this energy argument was originally developed for Brans–Dicke and for Kaluza–Klein theories is not surprising, owing to the fact that Brans–Dicke theory can be derived from a Kaluza–Klein theory with n extra dimensions and Brans–Dicke parameter $\omega = -(n-1)/n$ (Jordan 1959; Brans and Dicke 1961; Bergmann 1968; Wagoner 1970; Harrison 1972; Belinskii and Kalatnikov 1973; Freund 1982; Gross and Perry 1983; Cho 1992); this derivation is seen as a motivation for Brans–Dicke theory, and provides a useful way of generating exact solutions in one theory from known solutions in the other (Billyard and Coley 1997). Despite this derivation from Kaluza–Klein theory, the Jordan frame Brans–Dicke theory is sometimes considered in $D > 4$ spacetime dimensions (e.g. Majumdar 1997).

An independent argument supporting the choice of the Einstein frame as the physical one is obtained by considering (Cho 1992; Damour and Nordvedt 1993*a,b*) the linearized version of the theory. In the Jordan frame the metric is $\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (where $\eta_{\mu\nu}$ is the Minkowski metric), while in the Einstein frame the conformally transformed metric is

$$\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} \exp\left(\sqrt{\frac{16\pi G}{2\omega+3}}\,\tilde{\phi}\right) \simeq \eta_{\mu\nu} + \rho_{\mu\nu} \ , \tag{4.2}$$

where

$$\rho_{\mu\nu} = h_{\mu\nu} + \left(\sqrt{\frac{16\pi G}{2\omega+3}}\,\tilde{\phi}\right)\eta_{\mu\nu} \ . \tag{4.3}$$

The canonical action for a spin 2 field is not obtained from the metric $h_{\mu\nu}$, but it is

instead given by $\rho_{\mu\nu}$, and the spin 2 gravitational field is described by the Einstein frame corrections $\rho_{\mu\nu}$ to the flat metric. The Jordan frame corrections $h_{\mu\nu}$ to $\eta_{\mu\nu}$ describe a mixture of spin 0 and spin 2 fields (the fact that spin 0 and spin 2 modes are mixed together can also be seen from the full equations of motion of the theory).

A third argument has been proposed against the choice of the Jordan frame as the physical one: when quantum corrections are taken into account, one cannot maintain the minimal coupling of ordinary (i.e. other than the dilaton) matter to the Jordan metric (Cho 1997). This nullifies the traditional statement that the Jordan frame is to be preferred because the scalar couples minimally to all forms of matter in this frame.

These results are of the outmost importance for the experiments aimed at testing Einstein's theory: the Jordan frame versions of alternative classical theories of gravity are simply nonviable. However, despite the necessity of formulating Brans–Dicke theory in the Einstein frame, the classical tests of gravity for this theory are studied only for the Jordan frame formulation. In general, the authors working on the experimental tests of general relativity and alternative gravity theories do not seem to be aware of this paradoxical situation (e.g. Reasenberg *et al.* 1979; Will 1993).

The conformal rescaling has been used as a mathematical technique to generate exact solutions of Brans–Dicke theory from known solutions of the Einstein equations (Harrison 1972; Belinskii and Kalatnikov 1973; Lorentz–Petzold 1984) and approximate solutions in the linearized theory (Barros and Romero 1998).

(Generalized) scalar–tensor theories: This class of theories (Bergmann 1968; Wagoner 1970; Nordvedt 1970; Will 1993) is described by the Lagrangian density

$$S = \int d^4x\sqrt{-g}\left[f(\phi)R - \frac{\omega}{2}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi)\right] + S_{matter}\ , \tag{4.4}$$

where $\omega = \omega(\phi)$ and $V = V(\phi)$ (or by the more general action (2.15)). The corresponding field equations are

$$\begin{aligned} f(\phi)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) &= \frac{1}{2}T_{\mu\nu} + \frac{\omega}{2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right) \\ &\quad + \frac{1}{2}g_{\mu\nu}\left[Rf(\phi) - 2V\right] + \nabla_\mu\nabla_\nu f - g_{\mu\nu}\Box f\ , \end{aligned} \tag{4.5}$$

$$\Box\phi + \frac{1}{\omega}\left(\frac{1}{2}\frac{d\omega}{d\phi}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{df}{d\phi} - \frac{dV}{d\phi}\right) = 0\ , \tag{4.6}$$

where $T_{\mu\nu} = 2(-g)^{-1/2}\delta\mathcal{L}_{matter}/\delta g^{\mu\nu}$. The action (4.4) contains Brans–Dicke theory (2.1) and the nonminimally coupled scalar field theory (2.7) as particular cases. Theories with

more than one scalar field have also been investigated (Damour and Esposito–Farèse 1992; Berezin *et al.* 1989; Rainer and Zuhk 1996). A revival of interest in scalar–tensor theories was generated by the fact that in supergravity and superstring theories, scalar fields are associated to the metric tensor field, and that a coupling between a scalar field and gravity seems unavoidable in string theories (Green, Schwarz and Witten 1987). Indeed, scalar fields have been present in relativistic gravitational theories even before general relativity was formulated (see Brans 1997 for an historical perspective).

The necessity of the conformal transformation to the Einstein frame has been advocated in (Cho 1992, 1997) by investigating which linearized metric describes the physical spin 2 gravitons. A similar argument was presented in (Damour and Nordvedt 1993*a,b*), although these authors did not see it as a compelling reason to select the Einstein frame as the physical one. It has also been pointed out (Teyssandier and Tourrenc 1983; Damour and Esposito–Farèse 1992; Damour and Nordvedt 1993*a,b*) that the mixing of $g_{\mu\nu}$ and ϕ in the Jordan frame equations of motion makes the Jordan frame variables an inconvenient set for formulating the Cauchy problem. Moreover, the generalization to the case of tensor–multi scalar theories of gravitation, where several scalar fields instead of a single one appear, is straightforward in the Einstein frame but not so in the Jordan frame (Damour and Esposito–Farèse 1992). In the Einstein frame, ordinary matter does not obey the conservation law $\tilde{\nabla}^\nu \tilde{T}_{\mu\nu} = 0$ (with the exception of a radiative fluid with $\tilde{T} = 0$, which is conformally invariant) because of the coupling to the dilaton ϕ. Instead, the equation

$$\tilde{\nabla}_\nu \tilde{T}^{\mu\nu} = -\frac{1}{\Omega}\frac{\partial \Omega}{\partial \phi}\,\tilde{T}\,\tilde{\nabla}^\mu \phi \tag{4.7}$$

is satisfied. The total energy–momentum tensor of matter plus the scalar field is conserved (see Magnano and Sokolowski 1994 for a detailed discussion of conservation laws in both conformal frames).

The phenomenon of the propagation of light through scalar–tensor gravitational waves and the resulting time–dependent amplification of the light source provide an example of the physical difference between the Jordan and the Einstein frame.. In the Jordan frame the amplification effect is of first order in the gravitational wave amplitude (Faraoni 1996*a*), while it is only of second order in the Einstein frame (Faraoni and Gunzig 1998*a*).

It is interesting to note that, while the observational constraints on the Brans–Dicke parameter ω is $\omega > 500$ (Reasenberg *et al.* 1979), Brans–Dicke theory in the Einstein frame is subject to the much more stringent constraint $\omega > 10^8$ (Cho 1997). However, since the Einstein frame is the physical one, it is not very meaningful to present constraints on the Jordan frame parameter ω. Other formal and physical differences occur

in the Jordan and the Einstein frame: the singular points $\omega \to \infty$ in the ω–parameter space of the Jordan frame correspond to a minimum of the coupling factor $\ln \Omega(\phi)$ in the Einstein frame (Damour and Nordvedt 1993*a,b*). Singularities of the scalar–tensor theory may be smoothed out in the Jordan frame, but they reappear in the Einstein frame and plague the theory again due to the fact that the kinetic terms are canonical and the energy conditions (which are crucial in the singularity theorems) are satisfied in the Einstein frame (Kaloper and Olive 1998).

In (Bose and Lohiya 1997), the quasi–local mass defined in general relativity by the recent Hawking–Horowitz prescription (Hawking and Horowitz 1996) was generalized to n–dimensional scalar–tensor theories. It was shown that this quasi–local mass is invariant under the conformal transformation that reduces the gravitational part of the scalar–tensor theory to canonical Einstein gravity. The result holds under the assumptions that the conformal factor $\Omega\,(\phi)$ is a monotonic function of the scalar field ϕ, and that a global foliation of the spacetime manifold with spacelike hypersurfaces exists, but it does not require asymptotic flatness. Conformal invariance of the quasi–local mass was previously found in another generalization to scalar–tensor theories of the quasi–local mass (Chan, Creighton and Mann 1996).

The conformal transformation technique has been used to derive new solutions to scalar–tensor theories from known solutions of Einstein's theory (Van den Bergh 1980, 1982, 1983*a,b,c,d*; Barrow and Maeda 1990).

Nonlinear gravitational theories: a yet more general class of theories than the previous one is described by the Lagrangian density

$$\mathcal{L} = f(R)\sqrt{-g} + \mathcal{L}_{matter} \ , \tag{4.8}$$

which generates the field equations

$$\left(\frac{df}{dR}\right) R_{\mu\nu} - \frac{1}{2}\, f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu \left(\frac{df}{dR}\right) + g_{\mu\nu} \Box \left(\frac{df}{dR}\right) = T_{\mu\nu} \ , \tag{4.9}$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{matter}}{\delta g^{\mu\nu}} \ . \tag{4.10}$$

It is claimed in (Magnano and Sokolowski 1994) that the Einstein frame is the only physical one for this class of theories, using the energy argument of Sec. 3. The idea underlying the proof is to expand the function $f(R)$ as

$$f(R) = R + aR^2 + \ \ldots \ , \qquad a > 0 \ , \tag{4.11}$$

and then prove a positive energy theorem in the Einstein frame and the indefiniteness of the energy sign in the Jordan frame. The occurrence of singularities in higher order theories of gravity of the form (4.8) has been studied in (Barrow and Cotsakis 1988; Miritzis and Cotsakis 1996; Kaloper and Olive 1998), both in the Jordan and in the Einstein frame.

Kaluza–Klein theories: In classical Kaluza–Klein theories (Appelquist, Chodos and Freund 1987; Bailin and Love 1987; Overduin and Wesson 1997), the scalar field (dilaton) has a geometrical origin and corresponds to the scale factor of the extra spatial dimensions. The extra dimensions manifest themselves as matter (scalar fields) in the 4–dimensional spacetime. In the simplest version of the theory with a single scalar field[2], one starts with the $(4+d)$–dimensional action of vacuum general relativity

$$\hat{S} = \frac{1}{16\pi G} \int d^{(4+d)}x \left(\hat{R} + \hat{\Lambda}\right) \sqrt{-\hat{g}} \, , \tag{4.12}$$

where a caret denotes higher–dimensional quantities, the $(4+d)$–dimensional metric has the form

$$(\hat{g}_{AB}) = \begin{pmatrix} \hat{g}_{\mu\nu} & 0 \\ 0 & \hat{\phi}_{ab} \end{pmatrix} \, , \tag{4.13}$$

and $\hat{\Lambda}$ is the cosmological constant of the $(4+d)$–dimensional spacetime manifold. The latter is assumed to have the structure $M \otimes K$, where M is 4–dimensional and K is d–dimensional. Here the notations depart from those introduced at the beginning of this paper: the indices $A, B, \ ... \ = 0, 1, 2, 3, \ ... \ ,(4+d)$; $\mu, \nu, \ ... \ = 0, 1, 2, 3,$ and $a, b, ... = 4, 5, ..., (4+d)$. Dimensional reduction and the conformal transformation (1.1) with $\Omega = \sqrt{\phi}$, $\phi = \left|\hat{\phi}_{ab}\right|$, together with the redefinition of the scalar field

$$d\sigma = \frac{1}{2} \left(\frac{d+2}{16\pi G \, d} \right)^{1/2} \frac{d\phi}{\phi} \, , \tag{4.14}$$

leads to the Einstein frame action

$$S = \int d^4x \left[\frac{R}{16\pi G} - \frac{1}{2} \nabla_\mu \sigma \nabla^\mu \sigma - V(\sigma) \right] \sqrt{-g} \, , \tag{4.15}$$

[2]See e.g. (Berezin *et al.* 1989; Rainer and Zuhk 1996) for Kaluza–Klein theories with multiple dilatons.

$$V(\sigma) = \frac{R_K}{16\pi G} \exp\left(-\sqrt{\frac{16\pi G(d+2)}{d}}\,\sigma\right) + \frac{\hat{\Lambda}}{16\pi G} \exp\left(-\sqrt{\frac{16\pi G d}{d+2}}\,\sigma\right) \ , \qquad (4.16)$$

were R_K is the Ricci curvature of the metric on the submanifold K. Note that ϕ is dimensionless. However the redefined scalar field σ has the dimensions $[\sigma] = \left[G^{-1/2}\right]$, and is usually measured in Planck masses.

Unfortunately, the omission of a factor $1/\sqrt{16\pi G}$ in the right hand side of Eq. (4.14) seems to be common in the literature on Kaluza–Klein cosmology (cf. (Faraoni, Cooperstock and Overduin 1995) and footnote 11 of (Kolb, Salopek and Turner 1990)) and it leads to a non–canonical kinetic term $(16\pi G)^{-1}\nabla_\mu\sigma\nabla^\mu\sigma$ instead of $\nabla_\mu\sigma\nabla^\mu\sigma/2$ in the final action, and to a dimensionless field σ instead of one with the correct dimensions $[\sigma] = \left[G^{-1/2}\right]$. The error is perhaps due to the different notations used by particle physicists and by relativists; however insignificant it may appear to be, it profoundly affects the viability of the Kaluza–Klein cosmological model considered, since the spectral index of density perturbations is affected through the arguments of the exponentials in the scalar field potential (4.16) (Faraoni, Cooperstock and Overduin 1995). In the Jordan frame, the scalar field originating from the presence of the extra dimensions has kinetic energy that is negative definite or indefinite and an energy spectrum which is unbounded from below, implying that the ground state is unstable (Maeda 1986*a*; Maeda and Pang 1986; Sokolowski and Carr 1986; Sokolowski and Golda 1987). These defects are removed by the conformal rescaling (1.1) of the 4–dimensional metric. The requirement that the conformally rescaled system in 4 dimensions has positive definite energy (a purely classical argument) singles out a *unique* conformal factor. A proof of the uniqueness in 5–dimensional Kaluza–Klein theory was given in (Bombelli *et al.* 1987) and later generalized to an arbitrary number of extra spatial dimensions (Sokolowski 1989*a,b*).

From a quantum point of view, arguments in favour of the conformal rescaling have been pointed out (Maeda 1986*b*) and, in the context of 10– and 11–dimensional supergravity, the need for a conformal transformation in order to identify the physical fields was recognized (Scherk and Schwarz 1979; Chamseddine 1981; Dine *et al.* 1985). The requirement that the supersymmetry transformation of 11–dimensional supergravity take an $SU(8)$ covariant form leads to the same conformal factor (de Witt and Nicolai 1986). The conformal transformation which works as a cure for the dimensionally reduced $(4+d)$–dimensional Einstein gravity does not work for the dimensionally reduced Gauss–Bonnet theory (Sokolowski *et al.* 1991).

It is unfortunate that in the literature on Kaluza–Klein theories many authors neglected the conformal rescaling and only performed computations in the Jordan frame. Many results of classical Kaluza–Klein theories should be reanalysed in the Einstein frame (e.g. Alvarez and Belén Gavela 1983; Sadhev 1984; Deruelle and Madore 1987; Van den Bergh and Tavakol 1993; Fabris and Sakellariadou 1997; Kubyshin and Martin 1995; Fabris and Martin 1993; Chatterjee and Banerjee 1993; Biesiada 1994).

Torsion gravity: Theories of gravity with torsion have been studied in order to incorporate the quantum mechanical spin of elementary particles, or in attempts to formulate gauge theories of gravity (Hehl *et al.* 1976). An example is given by a theory of gravity with torsion, related to string theories, recently formulated both in the Jordan and in the Einstein frame (Hammond 1990, 1996). Torsion acts as a source of the scalar field; ordinary (i.e. different from the scalar field appearing in (1.9)) matter is added to the theory formulated in the Jordan or in the Einstein frame. This possibility differs from a conformal transformation of the total (gravity plus matter) system to the Einstein frame, and it does not appear to be legitimate since ordinary matter cannot be created as an effect of a conformal transformation. Although mathematically possible, this procedure appears to be very artificial, and it has been considered also in (Magnano and Sokolowski 1994) by including a nonminimal coupling of the scalar field to matter in the Jordan frame. The coupling was tuned in such a way that the Einstein frame matter is minimally coupled to the corresponding scalar field. The Jordan frame formulation of this theory is unviable because the large effects of the dilaton contradict the observations (Hammond 1996), and the Einstein frame version of this theory is the only possible option.

Induced gravity, which is described by the action

$$S = \int d^4x\sqrt{-g}\left[-\frac{\xi}{2}R\phi^2 - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right] , \tag{4.17}$$

is conformally invariant if $\xi = 1/6$. The field equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{1}{\xi\phi^2}\left[(1-4\xi)\,\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\left(2\xi - \frac{1}{2}\right)\nabla^\alpha\phi\nabla_\alpha\phi - Vg_{\mu\nu} + 2\xi g_{\mu\nu}\phi\Box\phi\right] , \tag{4.18}$$

$$\Box\phi - \xi R\phi - \frac{dV}{d\phi} = 0 . \tag{4.19}$$

Induced gravity with torsion in Riemann–Cartan spacetimes has been studied in (Park and Yoon 1997), and a generalization of the concept of conformal invariance has been

formulated.

Superstring theories: Although superstring theories are not classical theories of gravity, the effective action in the low energy limit is used to make predictions in the classical domain, and we comment upon this. The low–energy effective action for the bosonic string theory is given by (Callan *et al.* 1985)

$$S = \int d^{10}x\sqrt{-g}\left[\mathrm{e}^{-2\Phi}R + 4\nabla^{\mu}\Phi\nabla_{\mu}\Phi\right] + S_{matter}\ , \tag{4.20}$$

where Φ is the dimensionless string dilaton and the totally antisymmetric 3–form $H_{\mu\nu\lambda}$ appearing in the theory has been set equal to zero together with the cosmological constant (however, this is not always the case in the literature). By means of dimensional reduction and a conformal transformation, this model is reduced to 4–dimensional canonical gravity with two scalar fields:

$$\psi_1 = \frac{1}{\sqrt{16\pi G}}\left(6\ln b - \frac{\Phi}{2}\right)\ , \tag{4.21}$$

$$\psi_2 = \sqrt{\frac{3}{8\pi G}}\left(2\ln b + \frac{\Phi}{2}\right)\ , \tag{4.22}$$

where b is the radius of the manifold of the compactified extra dimensions. The action (4.20) has provided theoreticians with several cosmological models (Gasperini, Maharana and Veneziano 1991; Garcia–Bellido and Quiròs 1992; Gasperini and Veneziano 1992; Gasperini, Ricci and Veneziano 1993; Gasperini and Ricci 1993; Copeland, Lahiri and Wands 1994, 1995; Batakis 1995; Batakis and Kehagias 1995; Barrow and Kunze 1997). The issue of which conformal frame is physical in the low energy limit of string theories was raised in (Dick 1998).

5 Conformal transformations in cosmology

The standard big–bang cosmology based on general relativity is a very successful description of the universe that we observe, although cosmological solutions have been studied also in alternative theories of gravity. However, the need to solve the horizon, flatness and monopole problem, and to find a viable mechanism for the generation of density fluctuations evolving into the structures that we see today (galaxies, clusters, supeclusters and voids) motivated research beyond the big–bang model and led to the

idea of cosmological inflation (see Linde 1990; Kolb and Turner 1990; Liddle and Lyth 1993; Liddle 1996 for reviews). There is no universally accepted model of inflation, and several scenarios based either on general relativity or on alternative theories of gravity have been proposed. Since many of the alternative theories used involve a conformal transformation to a new conformal frame, it is natural that the problem of whether the Jordan or the Einstein frame is the physical one resurfaces in cosmology, together with the use of conformal rescalings to simplify the study of the equations of motion. It is possible that general relativity behaves as an attractor for scalar–tensor theories of gravity, and that a theory which departs from general relativity at early times in the history of the universe approaches general relativity during the matter–dominated era (Garcia–Bellido and Quirós 1990; Damour and Nordvedt 1993*a,b*; Mimoso and Wands 1995*a*; Oukuiss 1997) or even during inflation (Bekenstein and Meisels 1978; Garcia–Bellido and Quirós 1990; Barrow and Maeda 1990; Steinhardt and Accetta 1990; Damour and Vilenkin 1996) (unfortunately only the Jordan frame was considered in (Garcia–Bellido and Quirós 1990; Mimoso and Wands 1995*a*)). The convergence to general relativity cannot occur during the radiation–dominated era (Faraoni 1998).

One of the most important predictions of an inflationary scenario is the spectral index of density perturbations, which can already be compared with the observations of cosmic microwave background anisotropies and of large scale structures (Liddle and Lyth 1993). The spectral index is, in general, different in versions of the same scalar–tensor theory formulated in different conformal frames. For example, it is known that most classical Kaluza–Klein inflationary models based on the Jordan frame are allowed by the observations but are theoretically unviable (Sokolowski 1989*a,b*; Cho 1992) because of the energy argument discussed in Sec. 3; on the contrary, their Einstein frame counterparts are theoretically consistent but they are severely restricted or even forbidden by the observations of cosmic microwave background anisotropies (Faraoni, Cooperstock and Overduin 1995). In extended (La and Steinhardt 1989; Kolb, Salopek and Turner 1990; Laycock and Liddle 1994) and hyperextended (Steinhardt and Accetta 1990; Liddle and Wands 1992; Crittenden and Steinhardt 1992) inflation, differences between the density perturbations in the two frames have been pointed out (Kolb, Salopek and Turner 1990). The existing confusion on the problem of whether the Jordan or the Einstein frame is the physical one is particularly evident in the literature on inflation, and deeply affects the viability of the inflationary scenarios based on a theory of gravity which has a conformal transformation as an ingredient. Among these are extended (La and Steinhardt 1989; Laycock and Liddle 1994) and hyperextended (Kolb, Salopek and Turner 1990; Steinhardt and Accetta 1990; Liddle and Wands 1992; Crittenden and Steinhardt 1992) inflation, Kaluza–Klein (Yoon and Brill 1990; Cho and Yoon 1993; Cho 1994),

R^2–inflation (Starobinski 1980; Starobinski 1986; Maeda, Stein–Schabes and Futamase 1989; Liddle and Lyth 1993), soft and induced gravity inflation (Accetta, Zoller and Turner 1985; Accetta and Trester 1989; Salopek, Bond and Bardeen 1989). While several authors completely neglect the problem of which frame is physical, other authors present calculations in only one frame, and others again perform calculations in both frames, without deciding whether one of the two is physically preferred. Sometimes, the two frames are implicitely treated as if they both were simultaneously physical, and part of the results are presented in the Jordan frame, part in the Einstein frame. It is often remarked that all models of inflation based on a first order phase transition can be recast as slow–roll inflation using a conformal transformation (Kolb, Salopek and Turner 1990; Kalara, Kaloper and Olive 1990; Turner 1993; Liddle 1996), but the conformal rescaling is often performed without physical motivation. The justification for studying the original (i.e. prior to the conformal transformation) theory of gravity or inflationary scenario, which often relies on a specific theory of high energy physics, is then completely lost in this way. For example, one can start with a perturbatively renormalizable potential in the Jordan frame and most likely one ends up with a non–renormalizable potential in the Einstein frame. The conformal rescaling has even been used to vary the Jordan frame gravitational theory in order to obtain a pre–determined scalar field potential in the Einstein frame (Cotsakis and Saich 1994).

It is to be noted that the conformal transformation to a new conformal frame is sometimes used as a purely mathematical device to compute cosmological solutions by reducing the problem to a familiar (and computationally more convenient) scenario. The conformal transformation technique has been used to study also cosmological perturbations in generalized gravity theories (Hwang 1990; Mukhanov, Feldman and Brandenberger 1992; Hwang 1997*a*). This technique is certainly legitimate and convenient at the classical level, but it leads to problems when quantum fluctuations of the inflaton field are computed in the new conformal frame, and the result is mapped back into the "old" frame. Problems arise already at the semiclassical level (Duff 1981). This difficulty does not come as a surprise, since the conformal transformation introduces a mixing of the degrees of freedom corresponding to the scalar and the tensor modes. In general, the fluctuations in the two frames are physically inequivalent (Fujii and Nishioka 1990; Makino and Sasaki 1991; Nishioka and Fujii 1992; Fakir and Habib 1993; Fabris and Tossa 1997). There is ambiguity in the choice of vacuum states for the quantum fields: if a vacuum is chosen in one frame, it is unclear into what state the field is mapped in the other conformal frame, and one will end up, in general, with two different quantum states. The use of gauge–invariant quantities does not fix this problem (Fakir, Habib and Unruh 1992). The problem that plagues quantum fluctuations becomes relevant for

present–day observations because the quantum perturbations eventually become classical (Kolb and Turner 1990; Liddle and Lyth 1993; Tanaka and Sakagami 1997) and seed galaxies, clusters and superclusters.

Although the problem is not solved in general, the situation is not so bad in certain specific inflationary scenarios. In (Sasaki 1986; Makino and Sasaki 1991; Fakir, Habib and Unruh 1992), chaotic inflation with the quartic potential $V = \lambda\phi^4$ and nonminimal coupling of the scalar field was studied, and it was found that the amplitude of the density perturbations does not change under the conformal transformation. This result, however, relies on the assumption that one can split the inflaton field into a classical background plus quantum fluctuations (preliminary results when the decomposition is not possible have been obtained in (Nambu and Sasaki 1990)). Under slow–roll conditions in induced gravity inflation, the spectral index of density perturbations is frame–independent to first order in the slow–roll parameters (Kaiser 1995*a*). When the expansion of the universe is de Sitter–like, $a(t) \propto e^{Ht}$, $\dot{H} \approx 0$, it was found that the magnitude of the two–point correlation function is affected by the conformal transformation, but its dependence on the wavenumber, and consequently also the spectral index, is not affected (Kaiser 1995*b*). The spectral indices differ in the two conformal frames when the expansion of the scale factor is close to a power law[3] (Kaiser 1995*b*); often, workers in the field have not been sufficiently careful in this regard. Certain gauge–invariant quantities related to the cosmological perturbations turn out to be also conformally invariant under a mathematical condition satisfied by power law inflation and by the pole–like inflation encountered in the pre–big bang scenario of low energy string theory (Hwang 1997*b*).

At the level of the classical, *unperturbed* cosmological model, the occurrence of slow–roll inflation in the Einstein frame does not necessarily imply that inflation occurs also in the Jordan frame, or that it is of the slow–roll type, and the expansion law of the scale factor is in general different in the two conformal frames[4] (see Abreu, Crawford and Mimoso 1994 for an example).

Possible approaches to this problem are outlined in (Fakir and Habib 1993). Even if the same expansion law is achieved in the Jordan and the Einstein frame, the corresponding scalar field potentials can be quite different in the two frames. For example, power–law inflation is achieved by an exponential potential for a minimally coupled

[3]If inflation occurs in the early universe, it is not necessarily of the slow–roll type. The most well studied case of inflation without slow rolling is power law inflation which occurs for exponential potentials, obtained in almost all theories formulated in the Einstein frame.

[4]For extended inflation in Brans–Dicke theory with $\omega >> 1$, it has been proved that slow–roll inflation in the Einstein frame implies slow–roll inflation in the Jordan frame (but not viceversa) (Lidsey 1992; Green and Liddle 1996).

scalar field in the Einstein frame, and by a polynomial potential for its nonminimally coupled cousin in the Jordan frame (Abreu, Crawford and Mimoso 1994; Futamase and Maeda 1989; Faraoni 1996*b*).

Another cosmologically relevant aspect of the scalar field appearing in (1.9) is that it may contribute a significant fraction of the dark matter in the universe (Cho 1990; Cho and Yoon 1993; Delgado 1994; McDonald 1993*a,b*; Gasperini and Veneziano 1994; Gasperini 1994; Cho and Keum 1998). If one accepts the idea that the scalar field appearing in the expression for the conformal factor (1.9) is the field driving inflation (Salopek 1992; Cho 1992, 1994), then the inflationary scenario is completely determined. In fact, the conformal transformation to the Einstein frame in cosmology leads to either *a*) an exponential potential for the scalar field and to power–law inflation; *b*) a potential with more than one exponential term in Kaluza–Klein theories (Yoon and Brill 1990; Cho 1990; Cho and Yoon 1993), and to a kind of inflation that interpolates between power–law and de Sitter inflation (Easther 1994). It is also to be noted that, if a cosmological constant is present in a theory formulated in the Jordan frame, the new version of the theory in the Einstein frame has no cosmological constant (Collins, Martin and Squires 1989; Fujii 1998; Maeda 1992) but, instead, it exhibits an exponential term in the potential for the "new" scalar field.

The problem of whether a Noether symmetry is preserved by the conformal transformation has been analysed in (de Ritis *et al.* 1990; Demianski *et al.* 1991; Capozziello, de Ritis and Rubano 1993; Capozziello and de Ritis 1993; Capozziello, de Ritis and Marino 1997; Capozziello and de Ritis 1996, 1997*b*). The asymptotic evolution to an isotropic state of anisotropic Bianchi cosmologies in higher order theories with Lagrangian density of the form $\mathcal{L} = f(R)\sqrt{-g} + \mathcal{L}_{matter}$ was studied in (Miritzis and Cotsakis 1996) using the conformal rescaling as a mathematical tool. This study is relevant to the issue of cosmic no–hair theorems in these gravitational theories. In the Einstein frame, a homogeneous universe with matter satisfying the strong and dominant energy conditions and with a scalar field with a potential $V(\phi)$ locally convex and with zero minimum, can isotropize only if it is of Bianchi type I, V or VII. This result holds also in the Jordan frame if, in addition, the pressure of matter is positive (Miritzis and Cotsakis 1996).

6 Experimental consequences of the Einstein frame reformulation of gravitational theories

In most unified field theories, the conformal factor used in the conformal transformation is constructed using a physical field present in the gravitational theory (like a dilaton or a Brans–Dicke field) and therefore it is not surprising that it has certain physical effects which are, in principle, susceptible of experimental verification. The reality of the interaction with gravitational strength described by the dilaton was already stressed by Jordan (1949; 1955). The dilaton field in the Einstein frame couples differently to gravity and to matter (e.g. Horowitz 1990; Garfinkle, Horowitz and Strominger 1991), and the anomalous coupling results in a violation of the equivalence principle. Consider for example the action (4.4) plus a matter term in the Jordan frame: after the rescaling (1.1), (2.18), (2.24) has been performed, the scalar $\tilde{\phi}$ is minimally coupled to gravity, but it couples nonminimally to the other forms of matter via a field–dependent exponential factor:

$$S = \int d^4x \left\{ \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}_\mu \tilde{\phi} \right] + \mathrm{e}^{-\alpha\sqrt{G}\,\tilde{\phi}} \mathcal{L}_{matter} \right\} . \tag{6.1}$$

This leads to a violation of the equivalence principle which can, in principle, be tested by free fall experiments (Taylor and Veneziano 1988; Brans 1988; Cvetic 1989; Ellis *et al.* 1989; Cho and Park 1991; Cho 1992; Damour and Esposito–Farèse 1992; Cho 1994; Damour and Polyakov 1994*a,b*; Brans 1997). It is probably this anomalous coupling and the subsequent violation of the equivalence principle that explain the prejudice of many theoreticians against the use of the Einstein frame (which is not, however, a matter of taste, but is motivated by the independent energy arguments of Sec. 3). However, it is well known that although the Brans–Dicke scalar couples universally to all forms of ordinary matter in the Jordan frame, the *strong* equivalence principle is violated in this frame. This is sometimes understood as the fact that gravity determines a *local* value of the effective gravitational "constant" $G = \phi^{-1}$ (e.g. Brans 1997). In any case, the dilaton dependence of the coupling constants is to be regarded as an important prediction of string theories in the low energy limit, and as a new motivation for improving the present precision of tests of the equivalence principle.

By describing the gravitational interaction between two point masses m_1 and m_2 with the force law

$$F = \frac{G m_1 m_2}{r^2} \left(1 + \lambda \mathrm{e}^{-\mu r} \right) , \tag{6.2}$$

where λ and μ are, respectively, the strength and the range of the fifth force, one obtains

constraints on the range of these parameters. Due to the smallness of the values of λ allowed by the theory, the null results of the experiments looking for a fifth force still leave room for a theory formulated in the Einstein frame and with anomalous coupling (Cho 1992, 1994; Damour and Polyakov 1994*a,b*; Cho and Keum 1998).

There are also post–Newtonian effects and departures from general relativity in the strong gravity regime (Damour and Esposito–Farèse 1992), as well as differences in the gravitational radiation emitted and absorbed as compared to general relativity (Eardley 1975; Will and Eardley 1977; Will 1977; Will and Zaglauer 1989; Damour and Esposito–Farèse 1992).

If $\alpha(\phi) = \partial(\ln\Omega)/\partial\phi$, where Ω is the conformal factor in (1.1), then the post–Newtonian parameters γ and β (Will 1993) are given by (Damour and Esposito–Farèse 1992; Damour and Nordvedt 1993*a,b*)

$$\gamma - 1 = -\left.\frac{2\alpha^2}{1+\alpha^2}\right|_{\phi_0} , \tag{6.3}$$

$$\beta = 1 + \left.\frac{\alpha^2}{2\,(1+\alpha^2)^2}\;\frac{\partial^2(\ln\Omega)}{\partial\phi^2}\right|_{\phi_0} , \tag{6.4}$$

where $\phi_0 = \phi(t_0)$ is the value of the scalar field at the present time t_0, and it is assumed that the Brans–Dicke field only depends on time. The 1σ limits on γ from the Shapiro time delay experiment in the Solar System (Will 1993) are $|\gamma - 1| < 2 \cdot 10^{-3}$ (which implies $\alpha^2 < 10^{-3}$) and the combination $\eta \equiv 4\beta - \gamma - 3$ is subject to the constraint $|\eta| < 5 \cdot 10^{-3}$. By contrast, in a scalar–tensor theory, one expects $\alpha \approx 1$. This value of α could have been realistic early in the history of the universe with scalar–tensor gravity converging to general relativity at a later time during the matter–dominated epoch (Damour and Nordvedt 1993*a,b*). Accordingly, the Jordan and the Einstein frame would coincide today, the rescaling (1.1) differing from the identity only before the end of the matter–dominated era.

7 Nonminimal coupling of the scalar field

The material contained in this section is a summary of the state of the art on issues that have been only partially explored, results whose consequences are largely unknown, and problems that are still open. We try to point out the directions that, at present, appear most promising for future research. The reader should be aware of the fact that due to

the nature of such a discussion, the selection of topics presented here does not exhaust all the aspects involved.

The generalization to a curved spacetime of the flat space Klein–Gordon equation for a scalar field ϕ,

$$\Box\phi - \xi R\phi - \frac{dV}{d\phi} = 0 \,, \tag{7.1}$$

includes the possibility of an explicit coupling term $\xi R\phi$ between the field ϕ and the Ricci curvature of spacetime (Callan, Coleman and Jackiw 1970). There are many reasons to believe that a nonminimal (i.e. $\xi \neq 0$) coupling term appears: a nonminimal coupling is generated by quantum corrections even if it is absent in the classical action, or it is required in order to renormalize the theory (Freedman, Muzinich and Weinberg 1974; Freedman and Weinberg 1974). It has also been argued in quantum field theory in curved spaces that a nonminimal coupling term is to be expected whenever the spacetime curvature is large. This leads to what we will call the "ξ–problem", i.e. the problem of whether physics uniquely determines the value of ξ. The answer to this question is affirmative in many theories; several prescriptions for the coupling constant ξ exist and they differ according to the theory of the scalar field adopted. In general relativity and in all metric theories of gravity in which the scalar field ϕ has a non–gravitational origin, the value of ξ is fixed to the value 1/6 by the Einstein equivalence principle (Chernikov and Tagirov 1968; Sonego and Faraoni 1993; Grib and Poberii 1995; Grib and Rodrigues 1995; Faraoni 1996*b*). This is in contrast with a previous claim that nonminimal coupling spoils the equivalence principle (Lightman *et al.* 1975). However this claim has been shown to be based on flawed arguments; instead, it is the minimal coupling of the scalar field that leads to pathological behaviour (Grib and Poberii 1995; Grib and Rodrigues 1995). It is interesting that the derivation of the value $\xi = 1/6$ is completely independent of conformal transformations, the conformal structure of spacetime, the spacetime metric and the field equations for the metric tensor of the theory. The fact that the conformal coupling constant $\xi = 1/6$ emerges from these considerations is extremely unlikely to be a coincidence, but at present there is no satisfactory understanding of the reason why this happens, apart from the following naive consideration. No preferred length scale is present in the flat space massless Klein–Gordon equation and therefore no such scale must appear in the limit of the corresponding curved space massless equation when small regions of spacetime are considered, if the Einstein equivalence principle holds.

In all theories formulated in the Einstein frame, instead, the scalar field is minimally coupled ($\xi = 0$) to the Ricci curvature, as is evident from the actions (2.6), (2.11), (2.20), (2.25).

In many quantum theories of the scalar field ϕ there is a unique solution to the

ξ–problem, or there is a restricted range of values of ξ. If ϕ is a Goldstone boson in a theory with spontaneous symmetry breaking, $\xi = 0$ (Voloshin and Dolgov 1982). If ϕ represents a composite particle, the value of ξ should be fixed by the known dynamics of its constituents: for example, for the Nambu–Jona–Lasinio model, $\xi = 1/6$ in the large N approximation (Hill and Salopek 1992). In the $O(N)$–symmetric model with $V = \alpha\phi^4$, in which the constituents of the ϕ boson are scalars themselves, ξ depends on the coupling constants of the elementary scalars (Reuter 1994): if the coupling of the elementary scalars is $\xi_0 = 0$, then $\xi \in [-1, 0]$ while, if $\xi_0 = 1/6$, then $\xi = 0$. For Higgs scalar fields in the standard model and canonical gravity, the allowed range of values of ξ is $\xi \leq 0$, $\xi \geq 1/6$ (Hosotani 1985). The back reaction of gravity on the stability of the scalar ϕ in the potential $V(\phi) = \eta\phi^3$ leads to $\xi = 0$ (Hosotani 1985). The stability of a nonminimally coupled scalar field with the self–interaction potential

$$V(\phi) = \alpha\phi + m^2\phi^2/2 + \beta\phi^3 + \lambda\phi^4 - \Lambda \tag{7.2}$$

was shown to restrict the possible values of ξ and of the other parameters of this model (Bertolami 1987). Quantum corrections lead to a typical value of ξ of order 10^{-1} (Allen 1983; Ishikawa 1983). In general, in a quantum theory ξ is renormalized together with the other coupling constants of the theory and the particles' masses (Birrell and Davies 1980; Nelson and Panangaden 1982; Parker and Toms 1985; Hosotani 1985; Reuter 1994); this makes an unambiguous solution of the ξ–problem more difficult. In the context of cosmological inflation, a significant simplification occurs due to the fact that inflation is a classical, rather than quantum, phenomen: the energy scale involved is well below the Planck scale. The potential energy density of the inflaton field 50 e–folds before the end of inflation is subject to the constraint $V_{50} \leq 6 \cdot 10^{-11} m_{Pl}^4$, where m_{Pl} is the Planck mass (Kolb and Turner 1990; Turner 1993; Liddle and Lyth 1993). Moreover, the trajectory of the inflaton is peaked around classical trajectories (Mazenko, Unruh and Wald 1985; Evans and McCarthy 1985; Guth and Pi 1985; Pi 1985; Mazenko 1985*a,b*; Semenoff and Weiss 1985). Nevertheless, attempts have been made to begin the inflationary epoch in the context of string theory or quantum cosmology. A running coupling constant in inflationary cosmology was introduced in (Hill and Salopek 1992) and used to improve the chaotic inflationary scenario in (Futamase and Tanaka 1997). Asymptotically free theories in an external gravitational field described by the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left(aR^2 + b\,G_{GB} + c\,C_{\alpha\beta\gamma\delta}\,C^{\alpha\beta\gamma\delta} + \xi R\phi^2\right) + \mathcal{L}_{matter}\ , \tag{7.3}$$

where G_{GB} is the Gauss–Bonnet invariant, have a coupling constant $\xi(t)$ that depends on time and tends to $1/6$ when $t \to \infty$ (Buchbinder 1986; Buchbinder, Odintsov and

Shapiro 1986). In the renormalization group approach to grand unification theories in curved spaces it was found that, at the one loop level, $\xi(t) \to 1/6$ or $\xi(t) \to \infty$ exponentially (Buchbinder and Odintsov 1983, 1985; Buchbinder, Odintsov and Lichzier 1989; Odintsov 1991; Muta and Odintsov 1991; Elizalde and Odintsov 1994). However, this result is not free of controversies (Bonanno 1995; Bonanno and Zappalà 1997).

Nonminimal couplings of the scalar field have been widely used in cosmology, and therefore the above prescriptions have important consequences for the viability of inflationary scenarios. In fact, the nonminimal coupling constant ξ becomes an extra parameter of inflation, and it is well known that it affects the viability of many scenarios (Abbott 1981; Starobinsky 1981; Yokoyama 1988; Futamase and Maeda 1989; Futamase, Rothman and Matzner 1989; Amendola, Litterio and Occhionero 1990; Accioly and Pimentel 1990; Barroso *et al.* 1992; Garcia–Bellido and Linde 1995; Faraoni 1996*b*). The occurrence of inflation in anisotropic spaces is also affected by the value of ξ (Starobinsky 1981; Futamase, Rothman and Matzner 1989; Capozziello and de Ritis 1997*b*), which is relevant for the cosmic no–hair theorems. In many papers on inflation, the nonminimal coupling was used to improve the inflationary scenario; however, the feeling is that, in general, it actually works in the opposite direction (Faraoni 1997*a*). In some cases it may be possible to compare the spectral index of density perturbations predicted by the inflationary theory with the available observations of cosmic microwave background anisotropies in order to determine the value of ξ (Kaiser 1995*a*; Faraoni 1996*b*), or to obtain other observational constraints (Fukuyama *et al.* 1996). In cosmology, for chaotic inflation with the potential $V = \lambda\phi^4$, a nonminimal coupling to the curvature lessens the fine–tuning on the self–coupling parameter λ imposed by the cosmic microwave background anisotropies (Salopek, Bond and Bardeen 1989; Fakir and Unruh 1990*a,b*; Kolb, Salopek and Turner 1990; Makino and Sasaki 1991), $\lambda < 10^{-12}$. A nonminimal coupling term can also enhance the growth of density perturbations (Maeda 1992; Hirai and Maeda 1994; Hirai and Maeda 1997). For scalar fields in a Friedmann universe, the long wavelenghts λ do not scale with the usual reshift formula $\lambda/\lambda_0 = a(t)/a(t_0)$, but exhibit diffractive corrections if $\xi \neq 1/6$ (Hochberg and Kephart 1991).

The value of the coupling constant ξ affects also the success of the so–called "geometric reheating" of the universe after inflation (Bassett and Liberati 1998), which is achieved via a nonminimal coupling of the inflaton with the Ricci curvature, instead of the usual coupling to a second scalar field.

The "late time mild inflationary" scenario of the universe predicts very short periods of exponential expansion of the universe interrupting the matter era (Fukuyama *et al.* 1997). The model is based on a massive nonminimally coupled scalar field acting as dark matter. The success of the scenario depends on the value of ξ, and a negative sign of ξ

is necessary. However, the mechanism proposed in (Fukuyama *et al.* 1997) to achieve late time mild inflation turns out to be physically pathological from the point of view of wave propagation in curved spaces (Faraoni and Gunzig 1998*b*). At present, it is unclear whether alternative mechanisms can succesfully implement the idea of late time mild inflation.

The case $\xi \neq 0$ for a scalar field in the Jordan frame of higher dimensional models has been shown to have desirable properties in shrinking the extra dimensions (Sunahara, Kasai and Futamase 1990; Majumdar 1997), and has been used also for the Brans–Dicke field in generalized theories of gravity (Linde 1994; Laycock and Liddle 1994; Garcia–Bellido and Linde 1995). Exact solutions in cosmology have been obtained by using the conformal transformation (1.1), (2.9), (2.10) and starting from known solutions in the Einstein frame, in which the scalar field is minimally coupled (Bekenstein 1974; Froyland 1992; Accioly, Vaidya and Som 1983; Futamase and Maeda 1989; Abreu, Crawford and Mimoso 1994). From what we have already said in the previous sections, it is clear that, in general relativity with a nonminimally coupled scalar field, the Einstein and the Jordan frames are physically inequivalent but neither is physically preferred on the basis of energy arguments.

Nonminimal couplings of the scalar field in cosmology have been explored also in contexts different from inflation (Dolgov 1983; Ford 1987; Suen and Will 1988; Fujii and Nishioka 1990; Morikawa 1990; Hill, Steinhardt and Turner 1990; Morikawa 1991; Maeda 1992; Sudarsky 1992; Salgado, Sudarsky and Quevedo 1996, 1997; Faraoni 1997*b*) during the matter–dominated era (in the radiation–dominated era of a Friedmann–Lemaitre–Robertson–Walker solution, or in any spacetime with Ricci curvature $R = 0$, the explicit coupling of the scalar field to the curvature becomes irrelevant). In particular, a nonself–interacting, massless scalar field nonminimally coupled to the curvature with negative ξ has been considered as a mechanism to damp the cosmological constant (Dolgov 1983; Ford 1987; Suen and Will 1988) and solve the cosmological constant problem.

Another property of the nonminimally coupled scalar field is remarkable: while a big–bang singularity is present in many inflationary scenarios employing a minimally coupled scalar field, it appears that a nonminimally coupled scalar is a form of matter that can circumvent the null energy condition and avoid the initial singularity (Fakir 1998).

From the mathematical point of view, the action (2.7) is the only action such that the nonminimal coupling of ϕ to R involves only the scalar field but not its derivatives, and the coupling is characterized by a dimensionless constant (Birrell and Davies 1982). The Klein–Gordon equation arising from the action (2.7) is conformally invariant if $\xi = 1/6$ and $V(\phi) = 0$, or $V(\phi) = \lambda\phi^4$. Many authors choose to reason in terms of an effective

renormalization of the gravitational coupling constant

$$G_{eff} = \frac{G}{1 - 8\pi G\xi\phi^2} \ . \tag{7.4}$$

If $\phi = \phi(t)$, as in spatially homogeneous cosmologies or in homogeneous regions of spacetime, then the effective gravitational coupling $G_{eff} = G_{eff}(t)$ varies on a cosmological time scale. The possibility of a negative G_{eff} at high energies, corresponding to an antigravity regime in the early universe has also been considered (Pollock 1982; Novello 1982), also at the semiclassical level (Gunzig and Nardone 1984).

The solution of the ξ–problem is also relevant for the problem of backscattering of waves of the scalar ϕ off the background curvature of spacetime, and the creation of "tails" of radiation. If the Klein–Gordon wave equation (7.1) is conformally invariant, tails are absent in any conformally flat spacetime, including the cosmologically relevant case of Friedmann–Lemaitre–Robertson–Walker metrics (Sonego and Faraoni 1992; Noonan 1995). Other areas of gravitational physics for which the solution of the ξ–problem is relevant include the collapse of scalar fields (Frolov 1998), the theory of the structure and stability of boson stars (Van der Bij and Gleiser 1987; Liddle and Madsen 1992; Jetzer 1992), which is linked to inflation by the hypothesis that particles associated with the inflaton field may survive as dark matter in the form of boson stars. The ξ–problem is also relevant for the field of classical and quantum wormholes, in which negative energy fluxes are eliminated by restricting the allowed range of values of ξ (Ford 1987; Hiscock 1990; Coule 1992; Bleyer, Rainer and Zhuk 1994). Also the Ashtekar formulation of general relativity has been studied in the presence of nonminimally coupled scalar fields using a conformal transformation; the field equations in these variables are nonpolynomial, in contrast to the polynomial case of minimal coupling (Capovilla 1992).

8 Conclusions

Conformal transformations are extensively used in classical theories of gravity, higher–dimensional theories and cosmology. Sometimes, the conformal transformation is a purely mathematical tool that allows one to map complicated equations of motion into simpler equations, and constitutes an isomorphism between spaces of solutions of these equations. In this sense, the conformal transformation is a powerful solution–generating technique. More often, the conformal transformation to the Einstein frame is a map from a nonviable classical theory of gravity formulated in the Jordan frame to a viable one which, however, is not as well motivated as the starting one from the physical

perspective. A key role in establishing the viability of the Einstein frame version of the theory is played by the positivity of the energy and by the existence and stability of a ground state in the Einstein frame. It is to be remarked that the energy argument of Sec. 3 selecting the Einstein frame as the physical one is not applicable to quantum theories; in fact, the positivity of energy and the energy conditions do not hold for quantum theories. The weak energy condition is violated by quantum states (Ford and Roman 1992, 1993, 1995) and a theory can be unstable in the semiclassical regime (Witten 1982), or not have a ground state (e.g. Liouville's theory (D'Hoker and Jackiw 1982)).

Conformal transformations, nonminimal coupling, and the related aspects are important also for quantum and string theories (e.g. Stahlofen and Schramm 1989 – see Fulton, Rorlich and Witten 1962 for an early review) and for statistical mechanics (Dita and Georgescu 1989). For example, the conformal degree of freedom of a conformally flat metric has been studied in (Padmanabhan 1988) in order to get insight into the quantization of gravity in the particularly simple case when the spacetime metric is conformally flat: $g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$. In the context of quantum gravity, lower–dimensional theories of gravity have been under scrutiny for several years: when the spacetime dimension is 2 or 3, the metric has only the conformal degree of freedom (Brown, Henneaux and Teitelboim 1986), because the Weyl tensor vanishes and any two– or three–dimensional metric is conformally equivalent to the Minkowski spacetime of corresponding dimensionality (Wald 1984). The properties of the quantum–corrected Vlasov equation under conformal transformations have been studied in (Fonarev 1994). A nonminimal coupling of a quantum scalar field in a curved space can induce spontaneous symmetry breaking without negative squared masses (Madsen 1988; Moniz, Crawford and Barroso 1990; Grib and Poberii 1995). However, all these topics are beyond the purpose of the present work, which is limited to classical theories.

Many works that appeared and still appear in the literature are affected by confusion about the conformal transformation technique and the issue of which conformal frame is physical. Hopefully, these papers will be reanalysed in the near future in the updated perspective on the issue of conformal transformations summarized in this article. A change in the point of view is particularly urgent in the analysis of experimental tests of gravitational theories: most of the current literature refers to the Jordan frame formulation of Brans–Dicke and scalar–tensor theories, but it is the Einstein frame which has been established to be the physical one. A revision is also needed in the applications of gravitational theories to inflation; the predicted spectrum of density perturbations must be computed in the physical frame. In fact, only in this case it is meaningful to compare the theoretical predictions with the data from the high precision satellite experiments

which map the anisotropies in the cosmic microwave background – those already ongoing (*COBE* (Smoot *et al.* 1992; Bennet *et al.* 1996), and those planned for the early 2000s (NASA's *MAP* (MAP 1998) and ESA's *PLANCK* (PLANCK 1998)), and from the observations of large scale structures.

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